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Fundamental Study

Free shuffle algebras in language varieties [1]

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Abstract

We give simple concrete descriptions of the free algebras in the varieties generated by the "shuffle semirings" $\mathbf{L}_\Sigma := (P(\Sigma^*), +, \cdot, \otimes, 0, 1)$, or the semirings $\mathbf{R}_\Sigma := (R(\Sigma^*), +, \cdot, \otimes, ^*, 0, 1)$, where $P(\Sigma^*)$ is the collection of all subsets of the free monoid Σ^*, and $R(\Sigma^*)$ is the collection of all regular subsets. The operation $x \otimes y$ is the shuffle product.

Contents

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1. Introduction

In this paper we will give simple concrete descriptions of the free algebras in the varieties generated by the "shuffle semirings" of languages $\mathbf{L}_\Sigma := (P(\Sigma^*), +, \cdot, \otimes, 0, 1)$, for arbitrary sets Σ, where $P(\Sigma^*)$ is the collection of all subsets of the free monoid Σ^*. The operation $x \otimes y$ is the shuffle product. These descriptions can be used for several purposes: for establishing the decidability or undecidability of the corresponding equational theories; for finding equational (or other) complete axiomatizations, and for comparing different models of parallelism.

For years people have argued that the "interleaving model" of parallelism, usually modeled by the shuffle product of sets of words, is deficient. The fact that two processes a and b can run in parallel is better captured by the two-element poset $\{a, b\}$ with no nontrivial order relations. However, the results in this paper show that the set of equations valid for the interleaving model is the same as that for the poset model, at least when only the operations of parallel product, together with the "regular operations" of serial product, sum, and iterated serial product are involved. This fact contradicts the conclusion reached in a related paper by Gischer [10], that the "language model [of parallelism], which is based on interleaving is less general ... than the pomset model". The reason for this contradiction is explained by the difference between the ordering on pomsets used by Gischer, the subsumption order, and that used here. We also show how to use languages to represent certain pomsets, something that Pratt thought difficult. Quoting from [16], "We know of no satisfactory method of coding a set of posets as a set of strings, efficiently or otherwise, in a way that preserves algebraic properties such as concatenation." Of course, we do not code arbitrary sets of pomsets, or even arbitrary pomsets, but only certain *ideals* of serial-parallel pomsets, where the ordering on the pomsets is given in Definition 5.5. The new ordering on pomsets is one of the main contributions of this paper.

2. Preliminaries

Fix an "alphabet" Σ. The collection of all (regular) subsets of Σ^* forms an idempotent semiring where, for $a, b \subseteq \Sigma^*$,

$$\begin{aligned} a + b &:= a \cup b \\ a \cdot b &:= \{uv : u \in a,\ v \in b\} \\ 0 &:= \emptyset \\ 1 &:= \{\lambda\}, \end{aligned} \tag{1}$$

where λ denotes the empty word. We will consider the enrichment of this semiring by two operations: $a \mapsto a^*$ and $a, b \mapsto a \otimes b$, defined by

$$\begin{aligned} a^* &:= 1 + a + a^2 + \cdots \\ a \otimes b &:= \bigcup_{u \in a, v \in b} u \otimes v, \end{aligned} \tag{2}$$

where the "shuffle product" $\otimes$ is defined as follows on pairs of words.

$$\begin{aligned} \lambda \otimes u := u &= u \otimes \lambda \\ (xu) \otimes (yv) &:= x(u \otimes (yv)) + y((xu) \otimes v), \end{aligned}$$

for $x, y \in \Sigma$, $u, v \in \Sigma^*$.

The shuffle product is extended to *sets of words* pointwise: for $a, b \subseteq \Sigma^*$,

$$a \otimes b := \bigcup_{u \in a, v \in b} u \otimes v.$$

We will be considering (complete) semirings, as well as monoids enriched by a shuffle product.

Definition 2.1. A *bimonoid* $\mathrm{M} = (M, \cdot, \otimes, 1)$ consists of a monoid $(M, \cdot, 1)$ and a commutative monoid $(M, \otimes, 1)$. A *bimonoid morphism* $\mathrm{M} \to \mathrm{M}'$ is a function $M \to M'$ which preserves the unit and the two binary operations.

The only connection between the two monoids in a bimonoid is the common neutral element 1.

Note: We will occasionally abuse notation and write $x \in \mathrm{M}$, meaning $x \in M$.

Definition 2.2. An *ordered bimonoid* $(\mathrm{M}, \leqslant)$ is a bimonoid $\mathrm{M} = (M, \cdot, \otimes, 1)$ whose underlying set M is equipped with a partial ordering $\leqslant$ such that for all $x, y, a, b \in M$,

$$x \leqslant a, \ y \leqslant b \ \Rightarrow \ x \cdot y \leqslant a \cdot b \text{ and } x \otimes y \leqslant a \otimes b.$$

A *morphism of ordered bimonoids* is an order preserving bimonoid morphism.

Definition 2.3. A *shuffle semiring* $(S, +, \cdot, \otimes, 0, 1)$ is a bimonoid $(S, \cdot, \otimes, 1)$ enriched with a constant 0 and a commutative, associative, idempotent addition operation $+$, such that

$$\begin{aligned} x + 0 &= x \\ x \cdot 0 = 0 &= 0 \cdot x \\ x \otimes 0 &= 0 \\ x \cdot (y + z) &= (x \cdot y) + (x \cdot z) \\ (y + z) \cdot x &= (y \cdot x) + (z \cdot x) \\ x \otimes (y + z) &= (x \otimes y) + (x \otimes z), \end{aligned}$$

for all $x, y, z \in S$. A *morphism of shuffle semirings* is a function which preserves 0, 1 and the three binary operations $+, \cdot, \otimes$.

We define three shuffle semirings (indexed by the alphabet Σ):

$$\mathbf{L}_\Sigma := (P_\Sigma, +, \cdot, \otimes, 0, 1) \tag{3}$$

$$\mathbf{R}_\Sigma := (R_\Sigma, +, \cdot, \otimes, 0, 1) \tag{4}$$

$$\mathbf{F}_\Sigma := (F_\Sigma, +, \cdot, \otimes, 0, 1), \tag{5}$$

where P_Σ is the collection of all subsets, R_Σ consists of the regular subsets, and F_Σ consists of the finite subsets of Σ^*. The empty set is denoted 0 and the singleton set consisting of the empty word λ is 1. The addition operation in each is given by union; the operation $a \cdot b$ of complex concatenation is defined in (1). The reducts

$$\mathscr{L}_\Sigma := (P_\Sigma, \cdot, \otimes, 1)$$

$$\mathscr{R}_\Sigma := (R_\Sigma, \cdot, \otimes, 1)$$

$$\mathscr{F}_\Sigma := (F_\Sigma, \cdot, \otimes, 1)$$

of these structures are bimonoids, and $(\mathscr{L}_\Sigma, \subseteq)$, $(\mathscr{R}_\Sigma, \subseteq)$, and $(\mathscr{F}_\Sigma, \subseteq)$ are ordered bimonoids:

$$a \subseteq a',\ b \subseteq b' \ \Rightarrow\ (a \cdot b) \subseteq (a' \cdot b') \text{ and } (a \otimes b) \subseteq (a' \otimes b').$$

Definition 2.4. A *complete* shuffle semiring is a shuffle semiring having an infinitary operation $\sum_{i \in I} x_i$, for all sets I, such that $\sum_{i \in \{1,2\}} x_i = x_1 + x_2$, and such that

$$\sum_{i \in \emptyset} x_i = 0$$

$$\sum_{i \in I} x_i = x \text{ if } x_i = x \text{ for all } i \in I,\ I \neq \emptyset$$

$$y \cdot \left(\sum_{i \in I} x_i\right) = \sum_{i \in I} (y \cdot x_i)$$

$$\left(\sum_{i \in I} x_i\right) \cdot y = \sum_{i \in I} (x_i \cdot y)$$

$$y \otimes \left(\sum_{i \in I} x_i\right) = \sum_{i \in I} (y \otimes x_i)$$

$$\sum_{j \in J} \left(\sum_{i \in I_j} x_i\right) = \sum_{i \in I} x_i$$

where $I = \bigcup_{j \in J} I_j$ is the disjoint union of the sets I_j. A *morphism of complete shuffle semirings* is a shuffle semiring morphism which preserves the infinitary sums as well.

Note that $\mathbf{L}_\Sigma$ may be expanded to a complete shuffle semiring.

A *star shuffle semiring* $(S,+,\cdot,\otimes,^*,0,1)$ is a shuffle semiring enriched by a unary operation $^* : S \to S$, which need not satisfy any specific properties. A morphism of these structures is a shuffle semiring morphism which preserves the star operation. We will be considering the two star shuffle semirings of languages:

$$\mathbf{L}_\Sigma^* := (P_\Sigma,+,\cdot,\otimes,^*,0,1)$$
$$\mathbf{R}_\Sigma^* := (R_\Sigma,+,\cdot,\otimes,^*,0,1),$$

in which the star operation is given by (2).

3. Labeled posets

A Σ^**-labeled poset* $P = (|P|,\leqslant,\ell)$ consists of a poset $(|P|,\leqslant_P)$, sometimes written just $(|P|,\leqslant)$, and an assignment of a nonempty word $v\ell$ in Σ^* to each vertex v in P. (Here $|P|$ denotes the underlying set of elements or "vertices" of P, and we will sometimes write only P for this set. Thus the expression "$v \in P$" is meaningful.) When Σ is understood, we will say only "labeled poset". A *morphism* $f : P \to Q$ of Σ^*-labeled posets is a function $|P| \to |Q|$ which preserves the ordering and the labeling. We agree to identify isomorphic labeled posets, without further mention. (Many authors call an isomorphism class of a labeled poset a "pomset".) *To save space, we assume "poset" means "finite poset"*. We denote the empty poset by **1**. Two operations on (labeled) posets are important here, (sequential, or serial) product $P \cdot Q$ and shuffle (or parallel) product $P \otimes Q$. Given (labeled) posets P, Q, with $|P| \cap |Q| = \emptyset$,

$$P \cdot Q := (|P| \cup |Q|, \leqslant_{P\cdot Q})$$
$$P \otimes Q := (|P| \cup |Q|, \leqslant_{P\otimes Q}),$$

where for $v, v' \in |P| \cup |Q|$,

$$v \leqslant_{P\cdot Q} v' \iff v \leqslant_P v' \text{ or } v \leqslant_Q v' \text{ or } v \in |P| \text{ and } v' \in |Q|.$$
$$v \leqslant_{P\otimes Q} v' \iff v \leqslant_P v' \text{ or } v \leqslant_Q v'.$$

The labeling is extended to $P \otimes Q$ and $P \cdot Q$ in the obvious way. Note that the ordering $\leqslant_{P\otimes Q}$ is the disjoint union of the orderings on P and Q.

Definition 3.1. We let SP_{Σ^*}, for "series-parallel" or "shuffle product", denote the least class of posets containing the empty poset **1**, the singleton posets σ, labeled σ, for each $\sigma \in \Sigma^*$, closed under the operations $P \cdot Q$, $P \otimes Q$. The posets in SP_{Σ^*} will be called "series-parallel" posets. We let $\mathbf{SP}(\Sigma^*)$ be the following bimonoid:

$$\mathbf{SP}(\Sigma^*) := (\mathrm{SP}_{\Sigma^*},\cdot,\otimes,\mathbf{1}).$$

Let $P = (|P|,\leqslant)$ be a poset and suppose that v_1, v_2 are some vertices of P.

Definition 3.2. The vertex v_2 is an *immediate successor* of v_1, and v_1 is an *immediate predecessor* of v_2 if $v_1 < v_2$, and, for any $v \in |P|$, $v_1 \leqslant v < v_2 \Rightarrow v_1 = v$. Write *pred*$(v)$ for the set of all immediate predecessors of the vertex v, and *succ*(v) for the set of all immediate successors of v.

3.1. The bimonoid of A-labeled posets

Fix a set A. An *A-labeled poset* is an A^*-labeled poset such that the label of each vertex is a single letter in A.

The bimonoid of *all A-labeled posets* is denoted

$$\mathbf{Pos}(A) := (\mathrm{Pos}_A, \cdot, \otimes, \mathbf{1}).$$

Let SP_A denote the sub-collection of all series-parallel A-labeled posets, and let $\mathbf{SP}(A)$ denote the corresponding bimonoid. Identify the singleton poset labeled a with $a \in A$.

In this section, we will prove the following fact.

Theorem 3.3. $\mathbf{SP}(A)$ *is freely generated in the variety of all bimonoids by the set* A.

Proof. Let $\mathrm{M} = (M, \cdot, \otimes, 1)$ be any bimonoid, and let

$$h : A \to M$$

be a fixed function. We show how to extend h to a bimonoid morphism $h^{\#} : \mathbf{SP}(A) \to \mathrm{M}$. Let P be any poset in $\mathbf{SP}(A)$. If $P = \mathbf{1}$, then $Ph^{\#} := 1$; if $P = a$, for some $a \in A$, then $Ph^{\#} := ah$. Assume we have defined $Qh^{\#}$ on all posets in $\mathbf{SP}(A)$ with fewer than n elements, and assume that P has $n > 1$ elements. Any such poset can be written as either the serial product of at least two nonempty posets, or as the shuffle product of at least two nonempty posets; in the latter case, the expression is unique up to a permutation. If P is a shuffle product, write

$$P = Q_1 \otimes \cdots \otimes Q_k$$

where each poset Q_i cannot be written as a shuffle product of nonempty posets. Then we are forced to define

$$Ph^{\#} := Q_1 h^{\#} \otimes \cdots \otimes Q_k h^{\#}.$$

The value $Ph^{\#}$ is well-defined, due to the associativity and the commutativity of $\otimes$. If P is not a shuffle product of nonempty posets, write P as a serial product

$$P = Q_1 \cdot \ldots \cdot Q_k$$

where each poset Q_i cannot be written as a serial product of nonempty posets. Then we are forced to define

$$Ph^{\#} := Q_1 h^{\#} \cdot \ldots \cdot Q_k h^{\#}.$$

We show that the resulting function is a bimonoid morphism. Indeed, if

$$P = Q \otimes R$$

then

$$Qh^{\#} = Q_1 h^{\#} \otimes \cdots \otimes Q_k h^{\#},$$

$$Rh^{\#} = R_1 h^{\#} \otimes \cdots \otimes R_j h^{\#},$$

where each poset $Q_i, R_{i'}$ is $\otimes$-indecomposable. Thus, we may write

$$P = Q_1 \otimes \cdots \otimes Q_k \otimes R_1 \otimes \cdots \otimes R_j,$$

so that

$$\begin{aligned} Ph^{\#} &= Q_1 h^{\#} \otimes \cdots \otimes Q_k h^{\#} \otimes R_1 h^{\#} \otimes \cdots \otimes R_j h^{\#} \\ &= (Q_1 h^{\#} \otimes \ldots \otimes Q_k h^{\#}) \otimes (R_1 h^{\#} \otimes \cdots \otimes R_j h^{\#}) \\ &= Qh^{\#} \otimes Rh^{\#}, \end{aligned}$$

by the associativity and commutativity of $\otimes$. The argument that $(P \cdot Q)h^{\#} = Ph^{\#} \cdot Qh^{\#}$ is similar. Since the definition of $h^{\#}$ was forced, it is the only extension of h to a morphism. $\square$

Remark 3.4. A syntactic proof of Theorem 3.3 was given in [10]. Our proof is provided in order to make the paper more self-contained. Several operational semantics on bimonoid *terms* were considered in [1].

3.2. *Traces defined*

Recall that a *topological sort*, or *topological run* of a poset P is a bijection $s : [n] \to |P|$ such that

$$s_i \leqslant_P s_j \Rightarrow i \leqslant j,$$

where s_i is the value of s on $i \in [n]$. The notation $[n]$ denotes the set $\{1, 2, \ldots, n\}$. Suppose that (P, ℓ) is a Σ^*-labeled poset. Suppose that each vertex v of P which is labeled by a word $\sigma_1 \cdots \sigma_k$, $k = k_v \geqslant 1$, is replaced by the linearly ordered poset $v = v(1) < v(2) < \cdots < v(k_v)$, in which the label of the ith vertex $v(i)$ is σ_i. If the label of v is the empty word, the chain replacing v is $v(1)$, labeled by the empty word. (For example, if P is a 2-element poset $\{v_1, v_2\}$ in which the two elements v_1, v_2 are unrelated, and if $v_1\ell = abb$ and $v_2\ell = ba$, then the resulting poset has 5 elements: two disjoint chains, one of length 3 and one of length 2, labeled in the indicated way.) Call the resulting $\Sigma \cup \{\lambda\}$-labeled poset (P', ℓ') the *expansion* of (P, ℓ) (determined by the labeling $\ell : |P| \to \Sigma^*$).

Definition 3.5. The expansion (P', ℓ') of (P, ℓ) determined by the labeling ℓ is denoted P**Exp**. The ordering in the expansion of P is: $v(i) \leqslant v'(j)$ if either $v = v'$ and $1 \leqslant i \leqslant j \leqslant k_v$, or $v \neq v'$ and $v \leqslant_P v'$.

Definition 3.6. A *topological run* of a finite Σ^*-labeled poset P is a topological run of P**Exp**. A *trace of* P is a word

$$v_1\ell' v_2\ell' \ldots$$

formed by concatenating the letters labeling the vertices of a topological run of P**Exp**. The set of *all traces* of P is denoted **Tr**(P).

Remark 3.7. Suppose that $s : [n] \to P\mathbf{Exp}$ is a topological run of P**Exp**. Then, if $s_i = v(1)$ and $s_j = v'(1)$ and $v(1) < v'(1)$ in P**Exp**, then $i < j$. Thus, from any topological run of P**Exp** one can recover a topological run of P.

Remark 3.8. If P is a Σ^*-labeled poset such that the vertex v is labeled by the empty word, then **Tr**(P) is **Tr**(P'), where P' is obtained from P by deleting the vertex v from the set of vertices, and deleting all pairs $(v, v'), (v', v)$ from the ordering, considered as a set of ordered pairs.

We note the following properties of the function **Tr**.

Proposition 3.9.

$$\mathbf{Tr}(\mathbf{1}) = \{\lambda\}$$

$$\mathbf{Tr}(P \cdot Q) = \mathbf{Tr}(P) \cdot \mathbf{Tr}(Q)$$

$$\mathbf{Tr}(P \otimes Q) = \mathbf{Tr}(P) \otimes \mathbf{Tr}(Q).$$

Remark 3.10. By Theorem 3.3 and Proposition 3.9, $\mathbf{Tr} : \mathbf{SP}(\Sigma^*) \to \mathscr{L}_\Sigma$ is the unique bimonoid morphism which maps singletons labeled $w \in \Sigma^*$ to $\{w\}$.

3.3. *Traces of A^2-labeled posets*

In this section, we will be interested in certain kinds of labeled posets P. First, we assume that the alphabet can be written as the disjoint union of two sets, say A and $\overline{A}$, which are in bijective correspondence via a map

$$A \to \overline{A}$$
$$a \mapsto \overline{a}.$$

We further assume that each vertex v of P is labeled by a 2-letter word, say $a\overline{a}$, where the first letter of each label is $a \in A$ and the second letter $\overline{a} \in \overline{A}$ is the image of the first. Call such posets *A^2-labeled posets*.

The intuition behind such a labeling is a common one (see [2, 1, 9]): if u is the trace of a A^2-labeled poset, an occurrence of the letter a denotes the "initiation" of a process named a, and the matching occurrence of $\overline{a}$ denotes the "termination" of this process. Each occurrence of a letter in A has a matching occurrence in $\overline{A}$, as seen from the following lemma.

Lemma 3.11. *If a word u is a trace of an A^2-labeled poset P, then:*

1. $|u|_a = |u|_{\overline{a}}$, *for each $a \in A$.*
2. *If w is a prefix of u, $|w|_a \geqslant |w|_{\overline{a}}$, for each $a \in A$.*
3. *If distinct vertices in P have distinct labels,*

$$|u|_a \leqslant 1 \text{ and } |u|_{\overline{a}} \leqslant 1, \text{ for each } a \in A. \tag{6}$$

Here, $|u|_a$ denotes the number of occurrences of the letter a in the word u.

Definition 3.12. For any set A, let B_A, for "balanced", denote the set of all words satisfying the first two conditions of Lemma 3.11 as well as condition (6).

For the rest of this section, we will assume that distinct vertices have distinct labels, so that we may assume that the label $v\ell$ of the vertex v is the word $v\overline{v}$.

Definition 3.13. The preorder $u \sqsubseteq u'$ on the words in B_A is the least reflexive and transitive relation which satisfies the following conditions. For $u, u' \in B_A$, $u \sqsubseteq u'$ if, for some words u_1, u_2, and distinct letters $a, b \in A$,
(oo) $u = u_1abu_2$ and $u' = u_1bau_2$; or
(cc) $u = u_1\overline{ab}u_2$ and $u' = u_1\overline{ba}u_2$;
(co) $u = u_1\overline{a}bu_2$ and $u' = u_1b\overline{a}u_2$.

Notice that only the condition (co) in the definition of the preordering is asymmetric. If $X \subseteq B_A$, we say a word u is *maximal in* X if for any $u' \in X$,

$$u \sqsubseteq u' \Rightarrow u' \sqsubseteq u.$$

Hence, if u is maximal in X, no word obtained from u by replacing a subword $\overline{a}b$ of u by $b\overline{a}$ belongs to X. The converse is true modulo the rules (oo), (cc).

If $\mathbf{Tr}(P)$ is the set of all traces of an A^2-labeled poset P, then each word in $u \in \mathbf{Tr}(P)$ may be written as a product

$$u = s_0 p_1 s_1 \ldots p_{n-1} s_{n-1} p_n, \tag{7}$$

where each s_i is a word in A^+ and each p_j is a word in $\overline{A}^+$. (Recall that A^+ is the set of nonempty words on A.) We call words in A^+ *open words*, and words in $\overline{A}^+$ *closed words*. Since distinct vertices have distinct labels, we sometimes identify a closed or open word with the labels that occur in the word. A vertex v such that v but not $\overline{v}$ appears in the *prefix* u of a trace, is said to be *open in* u; if $\overline{v}$ appears, v is *closed* *in* u. Further, if $u \sqsubseteq u'$ and if $u' \in \mathbf{Tr}(P)$, then $u \in \mathbf{Tr}(P)$. In particular, if $v_1 v_2 \ldots v_n$ is a topological sort of the vertices of P, the word $v_1\overline{v_1}v_2\overline{v_2}\ldots v_n\overline{v_n}$ is a minimal word in $\mathbf{Tr}(P)$, where the label of v_i is $v_i\overline{v_i}$. (The names of the rules in Definition 3.13 are meant to suggest "open–open", "closed–closed" and "closed–open".)

We characterize the maximal words in $\mathbf{Tr}(P)$.

Proposition 3.14. *Suppose that P is a nonempty A^2-labeled poset such that distinct vertices have distinct labels. Then a word $u = s_0 p_1 s_1 \ldots s_{n-1} p_n \in \mathbf{Tr}(P)$, $n \geqslant 1$, is maximal iff*

s_0 is an open word listing all minimal vertices in P;

p_n is a closed word listing all maximal vertices in P;

($*$) *if vertex v is listed in the open word s_i, and vertex v' is listed in the closed word p_i, then $v' \in pred(v)$.*

Remark 3.15. The first two conditions follow from condition ($*$).

Proof of Proposition 3.14. First suppose that $u = s_0 p_1 s_1 \ldots s_{n-1} p_n \in \mathbf{Tr}(P)$ is maximal. Suppose that v is listed in the open word s_i, and vertex v' is listed in the closed word p_i. By applying the interchange rules (oo) and (cc) in Definition 3.13, we may assume that $\overline{v'}$ and v are adjacent in u, i.e., that $u = u_1\overline{v'}\,v u_2$, for some words u_1, u_2. If v' is not an immediate predecessor of v, then either v' is less than an immediate predecessor of v or v' is incomparable with v. In the latter case, $u_1 v\,\overline{v'} u_2$ is also a trace of P, so that u is not maximal. In the former case, suppose v' is less than a vertex $v'' \in pred(v)$. Then $\overline{v''}$ must occur in a word p_j, for some $j \leqslant i$ in order that u be a trace. But then u is not a trace, or $\overline{v'}$ should have occurred in some p_k, with $k < j \leqslant i$.

Now assume that $u = s_0 p_1 s_1 \ldots s_{n-1} p_n \in \mathbf{Tr}(P)$ satisfies condition ($*$) of Proposition 3.14. Then, if $u = u_1\overline{v'}v u_2$, for some vertices v', v, it follows that $v' \in pred(v)$, so that $u_1 v\overline{v'} u_2$ is not a trace. Hence u is maximal in $\mathbf{Tr}(P)$. $\square$

Definition 3.16. A word $u = s_0 p_1 s_1 \ldots p_{n-1} s_{n-1} p_n$ in $\mathbf{Tr}(P)$ is a *distinguishing trace for P* if the following strengthening of condition ($*$) in Proposition 3.14 holds. For each $0 < i < n$, for each vertex v in the open word s_i, a vertex v' is listed in the closed word p_i iff $v' \in pred(v)$. (Thus, all vertices listed in s_i have the same predecessors. It follows that each vertex closed in p_i has the set of vertices listed in s_i as its successors.)

If $P = \mathbf{1}$, $\mathbf{Tr}(P) = \{\lambda\}$, and in this case we say λ is distinguishing. Indeed, there is only one A^2-labeled poset with trace λ.

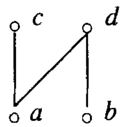
Corollary 3.17. *If u is a distinguishing trace for P, then u is maximal in* $\mathbf{Tr}(P)$.

A poset can be reconstructed from any distinguishing trace.

Lemma 3.18. *Suppose that P and Q are A^2-labeled posets satisfying the assumption of Proposition* 3.14. *If P and Q have the same distinguishing trace, P and Q are isomorphic.*

Not all (finite) A^2-labeled posets have distinguishing traces.

Example 3.19. Let P be the four-element set $\{a,b,c,d\}$ ordered so that the only non-trivial order relations are given by the equations $succ(a) = \{c,d\}$ and $succ(b) = \{d\}$.



Hence, a,b are minimal and c,d are maximal. There is one word which is maximal (up to commutativity inside open and closed words):

$$ab\bar{a}c\bar{b}d\bar{c}\bar{d}.$$

However, this word is not distinguishing for P, since the vertex a is not listed in the closed word preceding d.

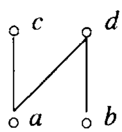
Definition 3.20. Suppose that P is an A^2-labeled poset such that distinct vertices have distinct labels. We say that P is *traceable* if there is a distinguishing trace in $\mathbf{Tr}(P)$.

We characterize traceability by a local property.

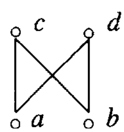
Definition 3.21. A poset P has the *zig-zag property* if whenever $c,d \in succ(a)$ and $b \in pred(d)$, it follows that $b \in pred(c)$, for any vertices $a,b,c,d \in P$.

Remark 3.22. The meaning of the term "zig-zag" here has nothing to do with its use in [17].

In graphical terms, the condition in Definition 3.21 can be restated as follows.



implies



Proposition 3.23. *The following conditions are equivalent for a poset P:*

1. *P has the zig-zag property.*
2. *For any vertices v, v' in P,*

$$pred(v) \cap pred(v') \neq \emptyset \;\Rightarrow\; pred(v) = pred(v'). \tag{8}$$

3. *For any vertices v, v' in P,*

$$succ(v) \cap succ(v') \neq \emptyset \;\Rightarrow\; succ(v) = succ(v'). \tag{9}$$

The word obtained from a distinguishing trace for P by deleting the rightmost closed word (listing the maximal elements) will be called a *partial distinguishing trace* for

P. Thus, a poset has a distinguishing trace iff it has a partial distinguishing trace. We use this fact in our proof of the following theorem.

Theorem 3.24. *A poset P is traceable iff P has the zig-zag property.*

Proof. The condition is clearly necessary. Now suppose that P has the zig-zag property (where the label of the vertex v is the word $v\bar{v}$).

Note that for any vertex v,

$$\{v' : pred(v') = pred(v)\} = succ(pred(v)).$$

The sets $pred(v)$, $v \in |P|$, together with the subset *max* of all maximal elements, form a decomposition of the vertices of P into disjoint subsets; similarly, the collection of all subsets $succ(v)$, together with the minimal elements *min*, form a decomposition of P into disjoint subsets. Recall that the height of a vertex v in a poset is the length of a longest sequence $v_0 < v_1 \cdots < v_n = v$, where $v_{i-1} \in pred(v_i)$, $0 \leqslant i \leqslant n$. Clearly, if v is any vertex of height $k \geqslant 1$, then each vertex in $pred(v)$ has height $\leqslant k-1$. Thus, if P has the zig-zag property, then so does the sub-poset P_k of P which consists of all vertices of height at most k, for each $k \geqslant 0$.

We will construct a word in $\mathbf{Tr}(P)$ by stages. At stage k we will construct a partial distinguishing trace t_k for the sub-poset P_k. When k is the height of P, we may quit.

Stage $k = 0$: Define t_0 as an open word listing all minimal vertices in P.

Stage $k+1$: Now assume that the word t_k has been constructed. We describe an algorithm to obtain t_{k+1}.

```
Let t := t_k;
While there is some vertex of height k+1 not open in t
    Choose one such vertex v;
    Let p be a closed word listing all predecessors of v;
    Let s be an open word listing all vertices in succ(pred v);
    Let t := t.p.s;
Let t_{k+1} := t.
```

Note that for any vertex v of height $k+1$, the set $pred(v)$ is disjoint from each set $pred(v')$, where the height of v' is at most k; further, all vertices in $pred(v)$ have been opened, but not closed in t_k. Lastly, if v and v' have height $k+1$ but $v' \notin succ((pred(v)))$, then $pred(v) \cap pred(v') = \emptyset$, since P has the zig-zag property.

Thus, when k is the height of the poset P, the word t_k followed by a word closing any vertex remaining open is a trace of P. We now observe that this word determines the ordering of P, and thus is a partial distinguishing trace for P. Hence P is traceable. □

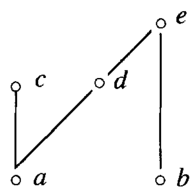
Remark 3.25. Note that traceability is a property of the underlying unlabeled poset.

Every series-parallel poset is traceable.

Lemma 3.26. *Every series-parallel poset has the zig-zag property.*

Proof. By induction on the number of operations needed to construct the poset from the singletons. □

Remark 3.27. The converse of this lemma is false. Let $|P| = \{a,b,c,d,e\}$ be the poset determined by the relations $succ(a) = \{c,d\}$, $succ(d) = succ(b) = e$.



Then P has the zig-zag property but is not series-parallel.

Corollary 3.28. *If P is a series-parallel A^2-labeled poset, such that each vertex is labeled with a distinct 2-letter word, P is traceable.*

Remark 3.29. The distinguishing traces have an equivalent description. Assume the poset P has the zig-zag property. Let D be the set of pairs (p,s) of sets of the form $p = pred(v)$, $s = succ(pred(v))$ ordered so that (p,s) is an immediate predecessor of (p',s') if $s \cap p' \neq \emptyset$. Note that if v is a minimal element, the pair $(\emptyset, min)$ is a pair in D; we also include the pair $(max, \emptyset)$ in D. In fact, $(\emptyset, min)$ is the minimum element and $(max, \emptyset)$ is the maximum element in D. We label the vertex (p,s) of D by the word $\overline{p}s$, where $\overline{p}$ is a closed word listing the vertices in p and where s is an open word listing the vertices in s. (When either p or s is the empty set, its label is λ.) If we now list the *labels* of the vertices of D in any topological order, we obtain a distinguishing trace for P.

We note the following converse of Corollary 3.17.

Proposition 3.30. *If P is traceable, then any maximal word in* $\mathbf{Tr}(P)$ *is a distinguishing trace for P.*

Proof. Let $u = s_0 p_1 s_1 \ldots p_n$ be any maximal trace in $\mathbf{Tr}(P)$. Suppose that v is a vertex listed in the open word s_i and that $v' \in pred(v)$. We must show that v' is listed in the closed word p_i. Now, since u is a trace, v' is listed in a closed word p_j for some $j \leqslant i$. In order to obtain a contradiction, suppose that $j < i$. Let v_1 be any vertex listed in the closed word p_i and let v_2 be any vertex listed in the open word s_j. Then, since u is maximal, by Proposition 3.14, $v' \in pred(v) \cap pred(v_2)$ and $v_1 \in pred(v)$. But, since P is traceable, $v_1 \in pred(v_2)$, so that u is not a trace. □

Remark 3.31. Theorem 4.1 in [1] is closely related to Corollary 3.28.

Remark 3.32. Grabowski [11] and Valdes [19] found the following "generalized zig-zag" characterization of series-parallel posets. A poset P is series-parallel iff there is no four-element subset $\{a,b,c,d\} \subseteq |P|$ whose only order relationships are given by

$$a < c, \quad a < d, \quad b < d.$$

A nice proof of this fact is given in [10]. We have found another characterization. Suppose that P is a traceable poset. Call a *zig-zag* in P an ordered pair of nonempty sets (L,U) of vertices such that for some $v \in U$, $L = pred(v)$ and $U = succ(pred(v))$. Suppose that (L_1,U_1) and (L_2,U_2) are zig-zags. We say that $L_1 < U_2$ if for some $v \in L_1$ there is a $v' \in U_2$ with $v < v'$. Similarly, for a vertex v, write $v < U$ (or $v > L$) if for some $v' \in U$, $v < v'$ (respectively, for some $v' \in L$, $v > v'$). Say a vertex v is *comparable* to the zig-zag (L,U) if either $v < U$ or $v > L$. The characterization is the following: A poset P is series-parallel iff P is traceable and whenever (L_1,U_1) and (L_2,U_2) are zig-zags with $L_1 < U_2$, then there is a zig-zag (L,U) such that:

- $L < U_2$ and $L_1 < U$.
- For any vertex v, if v is comparable to (L_i,U_i) for some $i = 1,2$, then v is comparable to (L,U).

A proof of this fact will be found in the Appendix.

4. An embedding theorem

In this section, we show that the bimonoid **SP**(A) of the series-parallel A-labeled posets belongs to the variety generated by the bimonoids $\mathscr{L}_\Sigma$. It then follows that **SP**(A) is the bimonoid freely generated by the set A in this variety, and that the variety of bimonoids is generated by the language bimonoids $\mathscr{L}_\Sigma$. In our argument, we make use of the results in the previous section.

Suppose that $h : \mathbf{SP}(A) \to \mathscr{L}_\Sigma$ is a bimonoid morphism extending a function which assigns a subset $L_a \subseteq \Sigma^*$ to each letter $a \in A$. The morphism h exists by Theorem 3.3, since $\mathscr{L}_\Sigma$ is a bimonoid.

We define the set $J = J(A,h) \subseteq A \times \Sigma^*$ as

$$J := \{(a,u) : u \in L_a\}. \tag{10}$$

Let

$$\pi_1 : \mathbf{SP}(J) \to \mathbf{SP}(A)$$

be the bimonoid morphism extending the projection $(a,u) \mapsto a$, and let

$$\pi_2 : \mathbf{SP}(J) \to \mathbf{SP}(\Sigma^*)$$

be the bimonoid morphism extending the projection $(a,u) \mapsto u$. Then, for any poset $P \in \mathbf{SP}(A)$, let

$$P_J := \pi_1^{-1}(P). \tag{11}$$

If P is any poset in $\mathbf{SP}(A)$, P_J is the collection of all posets obtainable from P by replacing an occurrence of the label a of any vertex by a letter $(a,u) \in J$, if $u \in L_a$. Different occurrences of the label a may be replaced by distinct letters (a,u), i.e., the word u depends on the occurrence. If $P' \in P_J$, $P'\pi_2$ is the poset obtained from P' by replacing the label (a,u) of any vertex by the word u in Σ^*.

Lemma 4.1.

$$Ph = \bigcup_{P' \in P_J} \mathbf{Tr}(P'\pi_2),$$

where the right-hand side is the union of the traces of the Σ^*-labeled posets $P'\pi_2$, for $P' \in P_J$.

Proof. We use induction on the structure of the poset P. If P is a singleton labeled a, then either $ah = \emptyset$ or not. If so, the union of the right-hand side is also empty, since the set P_J is empty. Otherwise, $Ph = L_a = \bigcup_{P' \in P_J} \mathbf{Tr}(P'\pi_2)$, since each poset $P'\pi_2$ is a singleton labeled by some word in L_a. If $P = Q \cdot R$, where Q, R are nonempty, the posets in P_J are all those which can be written as $Q' \cdot R'$, for some $Q' \in Q_J$, $R' \in R_J$. Thus,

$$\begin{aligned} Ph &= Qh \cdot Rh \\ &= \left(\bigcup_{Q' \in Q_J} \mathbf{Tr}(Q'\pi_2) \right) \cdot \left(\bigcup_{R' \in R_J} \mathbf{Tr}(R'\pi_2) \right) \\ &= \bigcup_{P' \in P_J} \mathbf{Tr}(P'\pi_2). \end{aligned}$$

The case that $P = Q \otimes R$ is similar. $\square$

Given the set A, let $\overline{A}$ be a set disjoint from A, and let

$$a \mapsto \overline{a}$$

be a bijection. Define the sets A_N and $\overline{A}_N$ by

$$\begin{aligned} A_N &:= A \times \mathbf{N} \\ \overline{A}_N &:= \overline{A} \times \mathbf{N} \end{aligned}$$

where $\mathbf{N} = \{0, 1, \ldots\}$. Denote elements in A_N as a_i, b_j, etc., and elements in $\overline{A}_N$ as $\overline{a}_i, \overline{b}_j$. Let $\Sigma(A)$ be the infinite alphabet

$$\Sigma(A) := A_N \cup \overline{A}_N.$$

The following theorem will have many applications, and is one of the main results of the paper.

Theorem 4.2. *Let $h_0 : \mathbf{SP}(A) \to \mathscr{L}_{\Sigma(A)}$ be the unique bimonoid morphism such that*

$$ah_0 = \{a_i\bar{a}_i : i \geqslant 0\}, \tag{12}$$

for each $a \in A$. Then h_0 is injective.

The proof requires some preliminary lemmas and definitions. By Lemma 4.1, for each poset P in $\mathbf{SP}(A)$, Ph_0 is the set of all traces of $(A_N)^2$-labeled posets obtainable by replacing the label a with some word $a_i\bar{a}_i$, in all possible ways.

The following fact is immediate.

Lemma 4.3. *If a word u is in the set Ph_0, then*
1. $|u|_{a_i} = |u|_{\bar{a}_i}$, *for each $a \in A$, $i \in \mathbf{N}$;*
2. *if w is a prefix of u, $|w|_{a_i} \geqslant |w|_{\bar{a}_i}$, for each $a \in A$.*

Note that a letter a_i may occur more than once in a word $u \in Ph_0$.

Definition 4.4. Let B denote the set of words on the alphabet $\Sigma(A)$ which satisfy the properties of Lemma 4.3.

We will extend the preorder of Definition 3.13. First we introduce the notion of an admissible endomorphism of $(A_N \cup \bar{A}_N)^*$.

Definition 4.5. A monoid endomorphism φ of $(A_N \cup \bar{A}_N)^*$ is *admissible* if for each $a \in A$, $i \in \mathbf{N}$, there is some $j \in \mathbf{N}$ with

$$a_i\varphi = a_j,$$
$$\bar{a}_i\varphi = \bar{a}_j.$$

Thus, admissible morphisms only change subscripts on letters. For example, an admissible morphism may identify a_1 and a_2.

Definition 4.6. The preorder $\sqsubseteq$ on B is the least reflexive and transitive relation which satisfies the following conditions. $u \sqsubseteq u'$ if, for some words w, w', and distinct letters a_i, b_j,

$$\begin{aligned}
&u = wa_ib_jw' && \text{and} \quad u' = wb_ja_iw'; \quad \text{or}\\
&u = w\bar{a}_i\bar{b}_jw' && \text{and} \quad u' = w\bar{b}_j\bar{a}_iw'; \quad \text{or}\\
&u = w\bar{a}_ib_jw' && \text{and} \quad u' = wb_j\bar{a}_iw'; \quad \text{or}\\
&u = u'\varphi, && \text{for some admissible } \varphi.
\end{aligned}$$

Note that if an admissible morphism φ is bijective, its inverse is also admissible, so that if $u' = u\varphi$, then $u' \sqsubseteq u$ and $u \sqsubseteq u'$.

Remark 4.7. Recall Definition 3.12 of the set B_A, for an arbitrary set A. Now, for words $u, u' \in B_{A_N}$, we have $u \sqsubseteq u'$ according to Definition 3.13 implies $u \sqsubseteq u'$ according to Definition 4.6.

As before, we say a word u is maximal in a set $X \subseteq B$ if whenever $u \sqsubseteq u'$, and $u' \in X$, then $u' \sqsubseteq u$.

Lemma 4.8. *Suppose that $P \in \mathbf{SP}(A)$ and $u \in Ph_0$ is maximal. Then u is a maximal trace in $\mathbf{Tr}(P')$, where P' is an A_N^2-labeled poset obtained from P by replacing the label of distinct vertices labeled a by distinct words $a_i\bar{a}_i$.*

Proof. We know by Lemma 4.1 that u is the trace of an A_N^2-labeled poset P' obtained by replacing any label $a \in A$ by a word $a_i\bar{a}_i$. Now suppose that two distinct vertices, say v and v' in P labeled a are labeled $a_i\bar{a}_i$ in P', and are labeled $a_i\bar{a}_i$ and $a_j\bar{a}_j$ in P'', where j is larger than any other subscript appearing in u. Let u' be the trace of P'' obtained from u by replacing the appropriate occurrences of a_i and $\bar{a}_i$ by a_j and $\bar{a}_j$, respectively. Then $u' \in Ph_0$ and $u \sqsubseteq u'$, since $u = u'\varphi$ where φ is the admissible endomorphism satisfying $a_j\varphi = a_i$, and otherwise is the identity. However, it is clearly not the case that $u' \sqsubseteq u$. Hence, u was not maximal in Ph_0. $\square$

Remark 4.9. The converse of Lemma 4.8 is true also. Suppose that $P \in \mathbf{SP}(A)$. If u is a maximal trace in $\mathbf{Tr}(P')$, where P' is an A_N^2-labeled poset obtained from P by replacing the labels of distinct vertices labeled a by distinct words $a_i\bar{a}_i$, then u is maximal in Ph_0.

Proof of Theorem 4.2. By Lemma 3.18, together with Lemma 3.30, if the maximal words in Ph_0 are the same as the maximal words in Qh_0, the two posets are isomorphic. Hence, if $Ph_0 = Qh_0$, P and Q are isomorphic. $\square$

We note the following fact concerning words in Ph_0 for later reference.

Proposition 4.10. *For each poset $P \in \mathbf{SP}(A)$, if $u \sqsubseteq u'$ and $u' \in Ph_0$, then $u \in Ph_0$.*

Definition 4.11. Let $\mathbf{Lg}$ be the variety of bimonoids generated by the collection of all algebras $\mathscr{L}_\Sigma = (P_\Sigma, \cdot, \otimes, 1)$.

The algebras $\mathscr{R}_\Sigma$ and $\mathscr{F}_\Sigma$ are also in $\mathbf{Lg}$, since they are subalgebras of one of the generating algebras.

The following result was obtained independently (by essentially the same argument) by Tschantz [18].

Corollary 4.12. *The bimonoid $\mathbf{SP}(A)$ belongs to the variety $\mathbf{Lg}$ generated by the bimonoids $\mathscr{L}_\Sigma$. Hence, the variety $\mathbf{Lg}$ is the variety of all bimonoids.*

Proof. Theorem 4.2 shows that **SP**(A) is isomorphic to a subalgebra of one of the generating algebras of the variety **Lg**, and hence belongs to **Lg**. It follows that every free bimonoid is in **Lg**, and hence every bimonoid is in **Lg**. □

The next corollary confirms a conjecture in [10].

Corollary 4.13. *The equational theory of the language bimonoids $\mathscr{L}_\Sigma$ is finitely axiomatizable.*

Proof. Since the axioms for **Lg** are precisely those axiomatizing bimonoids, six equations suffice. □

Definition 4.14. Let $\mathbf{Sh}_A \subseteq \mathscr{L}_{\Sigma(A)}$ denote the bimonoid which is the image of h_0. The sets of words in $\mathbf{Sh}_A$ form the least collection containing $\{\lambda\}$, and the languages

$$\{a_0\bar{a}_0, a_1\bar{a}_1, \ldots\},$$

for each $a \in A$, closed under complex concatenation and shuffle product.

Thus, $\mathbf{Sh}_A$ is another description of the free bimonoid in **Lg** generated by A. This description is rather cumbersome, since it is difficult to describe which subsets of $(A_N \cup \bar{A}_N)^*$ belong to $\mathbf{Sh}_A$.

It takes only a little more work to establish the following fact.

Theorem 4.15. *For each set A, **SP**(A) is in the variety generated by the bimonoids*

$$\mathscr{F}_\Sigma := (F_\Sigma, \cdot, \otimes, 1),$$

where F_Σ is the collection of all finite subsets of Σ^*.

Proof. For each $n \geqslant 1$, let $h_n : \mathbf{SP}(A) \to \mathscr{F}_{\Sigma(A)}$ be the unique bimonoid morphism mapping each letter $a \in A$ to the finite language

$$L_a^n := \{a_i\bar{a}_i : i = 0, \ldots, n-1\}. \tag{13}$$

By the argument in the proof of Theorem 4.2, if P and Q are two posets with at most n elements and if $Ph_n = Qh_n$, then P and Q are isomorphic. Thus, the morphism

$$\begin{aligned} \mathbf{SP}(A) &\to \prod_n \mathscr{F}_{\Sigma(A)} \\ P &\mapsto (Ph_n) \end{aligned}$$

is injective. □

Remark 4.16. For a letter $a \in A$, the *a-width* of a poset in **SP**(A) is defined inductively. The a-width of **1** is zero; the a-width of a is 1, and the width of the other singletons b, $b \neq a$ is 0; the a-width of $P \cdot Q$ is the maximum of the widths of P, Q; and the a-width of $P \otimes Q$ is the sum of the a-widths of P and Q. The *width of P* is the maximum of the a-widths of P, for $a \in A$. It can be shown that if P, Q are two posets in $\mathbf{SP}(A)$ of width at most n, then $Ph_n = Qh_n \Rightarrow P = Q$. It follows that the equational theory of bimonoids is decidable.

Corollary 4.17. $\mathbf{SP}(A)$ *is freely generated by the set A in both the variety generated by the bimonoids $\mathscr{F}_\Sigma$ and in the variety generated by the bimonoids $\mathscr{R}_\Sigma$ of regular sets. Thus, both of these varieties coincide with* $\mathbf{Lg}$.

5. The variety of ordered bimonoids

Recall from Definition 2.2 that an *ordered bimonoid* is a bimonoid whose underlying set is equipped with a partial ordering which is preserved by the bimonoid operations. If $f : A \to B$ is a morphism of ordered bimonoids, we say f is order-reflecting if $xf \leqslant yf \Leftrightarrow x \leqslant y$. (Thus, an order-reflecting morphism is also order-preserving.) Note that an order-reflecting morphism is necessarily injective. Say that the ordered bimonoid A is an ordered subalgebra of the ordered bimonoid B if there is an order-reflecting morphism $A \to B$. A *variety $\mathscr{V}$ of ordered bimonoids* is a collection of ordered bimonoids closed under products (ordered componentwise), ordered subalgebras and order-preserving morphic images. Equivalently (see [3]), a variety of ordered bimonoids is the collection of all ordered bimonoids which satisfy a set of inequations $t \leqslant t'$, for certain bimonoid terms t, t'. Any collection of ordered bimonoids is contained in a least variety of ordered bimonoids.

Definition 5.1. Let $\mathbf{Lg}_\leqslant$ denote the least variety of ordered bimonoids containing all of the ordered bimonoids of languages $(\mathscr{L}_\Sigma, \subseteq)$.

Example 5.2. The variety of all ordered bimonoids is axiomatized by inequations expressing the equations for bimonoids; e.g., the two inequations

$$x \cdot (y \cdot z) \leqslant (x \cdot y) \cdot z$$
$$(x \cdot y) \cdot z \leqslant x \cdot (y \cdot z)$$

express the associativity of the serial product operation.

Proposition 5.3. *The following inequalities hold in every ordered bimonoid* (M, $\leqslant$) *in* $\mathbf{Lg}_\leqslant$. *For all $a, b, c, d \in M$,*

$$(a \otimes b) \cdot (c \otimes d) \leqslant ac \otimes bd \tag{14}$$
$$ab \otimes ab \otimes ab \leqslant a(b \otimes b) \ \otimes \ (a \otimes a)b. \tag{15}$$

(We will assume that serial product binds more closely than shuffle, so that, e.g., $ac \otimes bd$ means $(ac) \otimes (bd)$.) In the bimonoids of languages $\mathscr{L}_\Sigma$, these inequations are most easily verified using the well-known characterization of the shuffle product by three monoid homomorphisms.

Corollary 5.4. *In any ordered bimonoid satisfying* (14),

$$\begin{aligned} ad &\leqslant a \otimes d \\ da &\leqslant a \otimes d \\ (a \otimes b) \cdot c &\leqslant ac \otimes b \\ a \cdot (b \otimes c) &\leqslant ac \otimes b. \end{aligned}$$

Proof. For the first inequality, put $b = c = 1$ in (14), giving $ad \leqslant a \otimes d$. Thus $da \leqslant a \otimes d$, since $d \otimes a = a \otimes d$. For the last two, first let $d = 1$ in (14). Then let $b = 1, d = b$. □

5.1. The ordered bimonoid of series-parallel posets

Recall the bimonoid morphism

$$h_0 : \mathbf{SP}(A) \to \mathscr{L}_{\Sigma(A)},$$

defined in (12), where $\Sigma(A) = A_N \cup \overline{A}_N$. Using h_0 and the preordering $\sqsubseteq$ of Definition 4.6, we will define an ordering on $\mathbf{SP}(A)$ and show that with this ordering, $\mathbf{SP}(A)$ is an ordered bimonoid. The intuition which suggested the definition of the ordering on $\mathbf{SP}(A)$ is that $P \leqslant Q$ if every trace of P is also a trace of Q, *no matter how the labels $a \in A$ are replaced by words on other alphabets*. However, it requires an argument to establish the correctness of this idea.

Note that each word $u \in Ph_0$ is a trace of a topological run of the poset P' obtained by replacing each vertex of P by a two-element chain; if in P, the vertex v was labeled a, then in P', the corresponding vertices, say $v(1), v(2)$, may be labeled $a_i, \overline{a}_i$, respectively, for some $i \geqslant 0$.

Definition 5.5. For A-labeled posets P, Q in $\mathbf{SP}(A)$, $P \leqslant Q$ iff for each word $u \in Ph_0$ there is a word $u' \in Qh_0$ with $u \sqsubseteq u'$.

We note some easy consequences of Definition 5.5. Recall definition (10), where $J = J(A, h_0)$.

Proposition 5.6. *Suppose that $P, Q \in \mathbf{SP}(A)$ and $P \leqslant Q$. Then*:

1. *P and Q have the same number of vertices. Moreover, for each letter $a \in A$, P and Q have the same number of vertices labeled a.*
2. *Suppose that $P' \in P_J$ and $u \in \mathbf{Tr}(P'\pi_2)$. Then there is some $Q' \in Q_J$ such that $u \in \mathbf{Tr}(Q'\pi_2)$.*
3. *For any topological run of P, there is a topological run of Q with the same trace.* □

Proposition 5.7. *For posets $P, Q \in \mathbf{SP}(A)$, the following are equivalent:*
1. $P \leqslant Q$.
2. $Ph_0 \subseteq Qh_0$.

Proof. Clearly, condition 2 implies condition 1. But if $P \leqslant Q$ and $u \in Ph_0$, it follows that $u \sqsubseteq u'$, for some $u' \in Qh_0$. Then by Lemma 4.1, u' is a trace of some $Q'\pi_2$, for some $Q' \in Q_J$. But then, by Proposition 4.10, u is also a trace of $Q'\pi_2$, showing $u \in Qh_0$. □

It now follows that the relation $\leqslant$ is a partial order on $\mathbf{SP}(A)$, since h_0 is injective.

Proposition 5.8. *Suppose that in $\mathbf{SP}(A)$, $P \leqslant P_1$ and $Q \leqslant Q_1$. Then*

$$P \cdot Q \leqslant P_1 \cdot Q_1 \quad \text{and} \quad P \otimes Q \leqslant P_1 \otimes Q_1.$$

Thus, $(\mathbf{SP}(A), \leqslant)$ is an ordered bimonoid in $\mathbf{Lg}_{\leqslant}$.

Proof. By Proposition 5.7,

$$P \cdot Q \leqslant P_1 \cdot Q_1 \;\Leftrightarrow\; (P \cdot Q)h_0 \subseteq (P_1 \cdot Q_1)h_0.$$

But

$$\begin{aligned}(P \cdot Q)h_0 &= Ph_0 \cdot Qh_0 \\ &\subseteq P_1h_0 \cdot Q_1h_0 \\ &= (P_1 \cdot Q_1)h_0.\end{aligned}$$

The argument for $\otimes$ is similar. Thus, $(\mathbf{SP}(A), \leqslant)$ is an ordered bimonoid, and $h_0 : \mathbf{SP}(A) \to \mathscr{L}_{\Sigma(A)}$ is an order-reflecting bimonoid morphism. Hence $\mathbf{SP}(A)$ belongs to $\mathbf{Lg}_{\leqslant}$. □

Remark 5.9. Suppose we define the following relation on posets in $\mathbf{SP}(A)$: $P \preceq Q$ if P and Q have the same number of vertices and there is an injective labeled poset morphism $Q \to P$. Then it is *not* the case that $Ph_0 \subseteq Qh_0$ implies $P \preceq Q$, although the converse does hold. For example, if one interprets the variable a in a special case $(a = b)$ of the inequation (15) as the one-element poset labeled a, it is not the case that

$$a^2 \otimes a^2 \otimes a^2 \preceq a(a \otimes a) \otimes (a \otimes a)a.$$

Gischer calls the order $\preceq$ the "subsumption ordering", and Pratt [16] denotes it by $\leqslant_\alpha$. See the Appendix, where a new proof is given of the result in [10], characterizing the ordered bimonoid $(\mathbf{SP}(A), \preceq)$.

Remark 5.10. Each of the conditions in Proposition 5.7 is equivalent to either of the following:
1. For each balanced word $u \in Ph_0$ there is a balanced word $w \in Qh_0$ with $u \sqsubseteq w$.
2. For each maximal word $u \in Ph_0$ there is a maximal word $w \in Qh_0$ with $u \sqsubseteq w$.

5.2. $(\mathbf{SP}(A), \leqslant)$ *is free in* $\mathbf{Lg}_{\leqslant}$

In this section, we show that the ordered bimonoid $(\mathbf{SP}(A), \leqslant)$ is the free ordered bimonoid in the variety of ordered bimonoids $\mathbf{Lg}_{\leqslant}$. In order to do so, it is necessary to prove the following fact.

Proposition 5.11. *Suppose that P, Q in $\mathbf{SP}(A)$. Then the following are equivalent:*
1. $P \leqslant Q$.
2. *For each alphabet Δ and each bimonoid morphism $g : \mathbf{SP}(A) \to \mathscr{L}_\Delta$, $Pg \subseteq Qg$.*

Proof. First, we show that condition 2 implies condition 1. Let $g = h_0$. Then $Ph_0 \subseteq Qh_0$, so that by Proposition 5.7, $P \leqslant Q$.

The proof that condition 1 implies condition 2 is longer. First, note that we may as well assume that $ag \neq \emptyset$, for all letters a which appear as labels of vertices of P, since otherwise $Pg = \emptyset \subseteq Qg$. For each integer $i \geqslant 0$, let Δ_i be the alphabet $\Delta \times \{i\}$, and let

$$\Delta' := \bigcup_{i \geqslant 0} \Delta_i.$$

We further assume that the alphabets Δ' and $\Sigma(A)$ are disjoint. Let $\kappa_i : \Delta^* \to (\Delta')^*$ be the unique monoid homomorphism such that

$$x \mapsto (x, i), \quad x \in \Delta.$$

Let g_1 be the bimonoid morphism $\mathbf{SP}A \to \mathscr{L}_{\Sigma(A) \cup \Delta'}$ determined by the map

$$A \to P_{\Sigma(A) \cup \Delta'}$$
$$a \mapsto \{a_i u \kappa_i \bar{a}_i : \ u \in ag, \ i \geqslant 0\}.$$

Let $p_0 : (\Sigma(A) \cup \Delta')^* \to \Sigma(A)^*$ and $p_1 : (\Sigma(A) \cup \Delta')^* \to \Delta^*$ be the two monoid homomorphisms determined by the conditions

$$xp_0 = \begin{cases} x \text{ if } x \in \Sigma(A) \\ \lambda \text{ if } x \in \Delta' \end{cases}$$

$$xp_1 = \begin{cases} y \text{ if } x = (y, i) \in \Delta' \\ \lambda \text{ if } x \in \Sigma(A). \end{cases}$$

Both p_0 and p_1 also denote the pointwise extension of these monoid homomorphisms to maps from subsets of $(\Sigma(A) \cup \Delta')^*$ to subsets of $\Sigma(A)^*$ and Δ^*, respectively. Thus,

$$g_1 \cdot p_0 = h_0$$
$$g_1 \cdot p_1 = g.$$

For any $(P, \leqslant, \ell) \in \mathbf{SP}(A)$, let $L(P) \subseteq (\Sigma(A) \cup \Delta')^*$ be the following set of words.

$$L(P) := \bigotimes_{v \in P} (v\ell) g_1.$$

Let

$$L(P)! := \{z \in L(P) : |z|_{a_i} \leqslant 1,\ a_i \in A_N\}.$$

Remark 5.12. If $P \leqslant Q$ in $\mathbf{SP}(A)$, then $L(P) = L(Q)$, by part 1 of Proposition 5.6.

Lemma 5.13. *Suppose that $z \in L(P)!$ and $zp_0 \in Ph_0$. Then $z \in Pg_1$.*

Proof. We assume without loss of generality that the vertices of P are nonnegative integers. Since $z \in L(P)!$, we assume that

$$z \in \bigotimes_{v \in P} a_v w_v \bar{a}_v,$$

where w_v is a word in Δ_v. We show that if zp_0 is in Ph_0, then z is a trace of the expansion $P\mathbf{Exp}$ of P determined by the labeling $v \in P \mapsto a_v w_v \bar{a}_v$. Since the words $\alpha_v := a_v w_v \bar{a}_v$ have no letter in common, we may identify the vertices of the expansion $P\mathbf{Exp}$ with the set of all nonempty prefixes of the words α_v ordered as follows: $u < u'$ if either u is a prefix of u' or u is a prefix of α_v, u' is a prefix of $\alpha_{v'}$ and $v < v'$ in P. The label of a nonempty prefix is its last letter. Now write the word z as a product of letters:

$$z = z_1 z_2 \ldots z_m.$$

For each letter z_i of z there is a unique vertex v of P such that z_i is a letter in the word α_v. Let $\pi(\alpha_v, i)$ denote the prefix of α_v determined by deleting from $z_1 \ldots z_i$ all letters not in the alphabet of α_v. Then z determines the following function:

$$\begin{aligned} s : [m] &\to P\mathbf{Exp} \\ i &\mapsto \pi(\alpha_v, i) \text{ if } z_i \text{ is a letter in } \alpha_v. \end{aligned}$$

Claim. *If $zp_0 \in Ph_0$, then s is a topological run of $P\mathbf{Exp}$ whose trace is z.*

Indeed, assume $i, j \in [m]$ and $is < js$. If is is a prefix of js, then clearly $i < j$. Otherwise, is is a prefix of α_v and js is a prefix of $\alpha_{v'}$ and $v < v'$ in P. But since $zp_0 \in Ph_0$, the letter $\bar{a}_v$ occurs before $a_{v'}$ in z. Thus, the entire word α_v will be listed by s before any prefix of $\alpha_{v'}$ is. Thus s is a topological run of $P\mathbf{Exp}$. The trace of this run is the word z, since for each $i \in [m]$, the value is is a prefix whose last letter is z_i. □

Lemma 5.14. *For any word $u \in Pg$ there is a word z in Pg_1 such that*

$$zp_1 = u.$$

Proof. Suppose that u is the trace of a topological run $s : [n] \to P\mathbf{Exp}$ of an expansion of P obtained by relabeling the vertex $v \in P$ by the word $w_v \in (v\ell)g$. If we consider the relabeling

$$v \mapsto a_v(w_v \kappa_v)\bar{a}_v,$$

where $a = v\ell$, we can modify the run s (in several ways) to obtain the desired word z. $\square$

We may now complete the proof of Proposition 5.11. Let $u \in Pg$. By Lemma 5.14, there is a word $z \in Pg_1 \cap L(P)!$ with $zp_1 = u$. But since $g \cdot p_0 = h_0$, $zp_0 \in Ph_0$. But since $P \leqslant Q$, $L(P) = L(Q)$ and $Ph_0 \subseteq Qh_0$. Thus, by Lemma 5.13, $z \in Qg_1$, so that

$$u = zp_1 \in Qg. \qquad \square$$

For later use, we note the following fact.

Corollary 5.15. *Suppose that B is any subset of* $\mathbf{SP}(A)$ *and* $g : \mathbf{SP}(A) \to \mathscr{L}_\Delta$ *is any bimonoid morphism. Then*

$$Ph_0 \subseteq \bigcup_{Q \in B} Qh_0 \Rightarrow Pg \subseteq \bigcup_{Q \in B} Qg.$$

Proof. Indeed, let u be a word in Pg. Then, by Lemma 5.14, there is a word $z \in Pg_1 \cap L(P)!$ with $zp_1 = u$. Thus, $zp_0 \in Qh_0$, for some $Q \in B$, so that $z \in Qg_1$ and $zp_1 = u \in Qg$. $\square$

The following theorem follows easily.

Theorem 5.16. $(\mathbf{SP}(A), \leqslant)$ *is the free ordered bimonoid in* $\mathbf{Lg}_\leqslant$*, freely generated by A.*

Proof. We have already shown that $(\mathbf{SP}(A), \leqslant)$ belongs to $\mathbf{Lg}_\leqslant$ in Proposition 5.8. In order to prove freeness, it is enough to prove that for any $\mathrm{M} = \mathscr{L}_\Sigma$ in $\mathbf{Lg}_\leqslant$ and any function $f : A \to M$, there is a unique extension of f to an order-preserving bimonoid morphism $\mathbf{SP}(A) \to \mathrm{M}$. But by Proposition 5.11, if g is the unique extension of f to an unordered bimonoid morphism, g is necessarily order-preserving. $\square$

Corollary 5.17. [9, Theorem 2.2] *A bimonoid morphism g :* $\mathbf{SP}(A) \to \mathscr{L}_\Sigma$ *such that ag is a set of 2-letter words, for each $a \in A$, is called a* diagram morphism. *Then, if $P, Q \in$* $\mathbf{SP}(A)$ *and $Pg \subseteq Qg$, for all diagram morphisms g, then $Ph \subseteq Qh$ for all bimonoid morphisms h :* $\mathbf{SP}(A) \to \mathscr{L}_\Sigma$.

Proof. Since one diagram morphism is h_0, the result follows from Proposition 5.11. $\square$

Remark 5.18. The argument given in [9, Theorem 2.2] in fact establishes Proposition 5.11, but the fact that h_0 is injective was not observed.

Remark 5.19. Using the same argument as for Corollary 4.17, it can be shown that $\mathbf{Lg}_\leqslant$ is generated by the ordered bimonoids $(\mathscr{R}_\Sigma, \subseteq)$ and by the ordered bimonoids $(\mathscr{F}_\Sigma, \subseteq)$.

Remark 5.20. Using Remark 4.16, it follows that the (in)equational theory of $\mathbf{Lg}_{\leqslant}$ is decidable.

Remark 5.21. Another representation of the free ordered bimonoid in $\mathbf{Lg}_{\leqslant}$ is the ordered bimonoid $(\mathbf{Sh}_A, \subseteq)$, where the ordering is set inclusion. Although the description of the languages in $\mathbf{Sh}_A$ is inconvenient, the order is quite natural. The representation of the free algebra by labeled posets may result in more efficient algorithms. The obvious algorithm to decide the validity of an identity in $\mathbf{Lg}_{\leqslant}$ using the language representation uses hyperexponential time, but using the poset representation, the algorithm is not worse than $O(n \log n)$. See Section 7.

Remark 5.22. There is an algorithm to produce the set of all words in Pg. The algorithm is nondeterministic. Given any word w in the set B_{A_N} (see Definition 3.12) and the languages $L_a := ag$, the algorithm produces a set of words in Δ^*. We then claim that when the words w range over all words in $B_{A_N} \cap Ph_0$, or just the maximal words in Ph_0, the set of words produced is precisely Pg.

The input to the algorithm is a word w in B_{A_N}, and, in addition, for each letter a_i (which we may assume appears in w), a word $u(a,i) \in \Delta^*$. There is also an "output word", `OW`, which is initialized to λ. At any moment a subset of the words $u(a,i)$ is "open". Initially this set is empty. During the course of the algorithm, the word denoted by $u(a,i)$ may change.

We let `WRITE` abbreviate the following nondeterministic procedure:

```
Do the following operation some finite number of times:
    If the set of open words is nonempty,
        Choose some nonempty open word, say
         u(a,j) = xu', x in Δ,u' ∈ Δ*.
        OW := OW.x
        u(a,j) := u'.
```

The algorithm is now the following:

```
Set the output word OW to the empty word;
Set the collection of open words to the empty set;
For i = 1 to length w do:
           begin:
           WRITE;
           If the i-th letter of w is a_j
                       Insert u(a,j) into the set of open words;
           If the i-th letter of w is ā_j
                       Set OW := OW.u(a,j)
                       Delete u(a,j) from the set of open words.
           end.
```

The set of all possible output words produced by this algorithm with input word $w \in B_A \cap Ph_0$, and $u(a,i) \in ag$ is Pg.

6. Shuffle semirings of languages

We now expand the language bimonoids we have been considering to include first the binary addition operation, then arbitrary sums, and then just the geometric sums given by the star operation. In each case, we are interested in the varieties generated by the language structures, and the free algebras in each of these varieties are described.

6.1. Closed subsets of free ordered bimonoids

We will outline a general adjunction result connecting shuffle semirings and ordered bimonoids. By means of a sequence of examples we show how this general result yields a description of the free shuffle semiring in the variety of shuffle semirings **Lang** generated by the structures $\mathbf{L}_\Sigma$.

We let **S** denote the category of all shuffle semirings and shuffle semiring morphisms. If $S = (S, +, \cdot, \otimes, 0, 1)$ is a shuffle semiring, its reduct $S\mathbf{O} = (S, \cdot, \otimes, \leqslant, 1)$ is an ordered bimonoid, where

$$x \leqslant y \iff x + y = y \iff x + z = y,$$

for some $z \in S$. If $h : S \to S'$ is a morphism of shuffle semirings, then $h = h\mathbf{O}$ is a morphism $S\mathbf{O} \to S'\mathbf{O}$. Thus $\mathbf{O}$ is a functor from the category of shuffle semirings to the category of ordered bimonoids.

Suppose that K is a class of shuffle semirings and $\mathbf{Var}[K]$ is the variety generated by K. The class $K\mathbf{O}$ consists of the ordered bimonoids $S\mathbf{O}$, for $S \in K$. Let $\mathscr{V}$ be the variety of ordered bimonoids generated by $K\mathbf{O}$.

Notation: Let K_L be the class of language shuffle semirings $\mathbf{L}_\Sigma$, for all alphabets Σ.

Example 6.1. If $K = K_L$, $\mathbf{Var}[K_L] = \mathbf{Lang}$ and $\mathscr{V} = \mathbf{Lg}_\leqslant$.

Let $F_K(A)$ be the ordered bimonoid freely generated by the set A in $\mathscr{V}$. Then, for each $a, b \in F_K(A)$: $a \leqslant b$ iff $ah \leqslant bh$, for all ordered bimonoid morphisms $h : F_K(A) \to S\mathbf{O}$, $S \in K$.

Example 6.2. When $K = K_L$, $F_K(A) = \mathbf{SP}(A)$. Further, we have the stronger result: for P, Q in $\mathbf{SP}(A)$,

$$P \leqslant Q \iff Ph_0 \leqslant Qh_0,$$

by Propositions 5.11 and 5.7.

Definition 6.3. Let $h : F_K(A) \to S\mathbf{O}$ be an ordered bimonoid morphism with $S \in K$. We extend h to a function from finite subsets of $F_K(A)$ to the underlying set of S. Suppose that B_0 is a finite subset of $F_K(A)$.

$$B_0h := \sum_{b \in B_0} bh.$$

Note that $B_0h = \sup_{b \in B_0} bh$. For any finite set $B_0 \subseteq F_K(A)$, let $cl(B_0)$ denote the set of all elements $b \in F_K(A)$ such that

$$bh \leqslant B_0h,$$

for every morphism $h : F_K(A) \to S\mathbf{O}$, $S \in K$. Note that $B_0 \subseteq cl(B_0)$, for any finite set B_0.

Definition 6.4. A set $B \subseteq F_K(A)$ is *closed* if

$$B = \bigcup_{B_0 \subseteq B,\, B_0 \text{ finite}} cl(B_0).$$

Thus, a set B is closed if for each finite subset B_0 of B, $cl(B_0) \subseteq B$.

Example 6.5. In view of the universal properties of h_0, a subset B of $\mathbf{SP}(A)$ is closed if for each finite set $B_0 \subseteq B$, and each $P \in \mathbf{SP}(A)$, $Ph_0 \subseteq B_0h \Rightarrow P \in B$.

We list some easy consequences of the definition.

Proposition 6.6. (a) *The empty set $\emptyset$ and $F_K(A)$ are closed.*

(b) *If B_j, $j \in J$, are closed, then so is the intersection $\bigcap_{j \in J} B_j$. Thus each set $B \subseteq F_K(A)$ is contained in a least closed set, namely*

$$\bigcup_{B_0 \subseteq B,\, B_0 \text{ finite}} cl(B_0),$$

which is denoted also by $cl(B)$.

(c) *Each closed set is an ideal; i.e., if B is closed and $b \leqslant c$ and $c \in B$, then $b \in B$.*

(d) *Each one-generated ideal in $F_K(A)$ is closed.*

(e) *If B and C are closed, then $B \subseteq C$ iff for each $b \in B$ there exists a finite set $C_0 \subseteq C$ such that $b \in cl(C_0)$.*

Definition 6.7. Let $\mathbf{I}_\omega(K;A)$ denote the set of all finitely generated closed subsets of $F_K(A)$:

$$\mathbf{I}_\omega(K;A) := \{cl(B_0) : B_0 \subseteq F_K(A),\ B_0 \text{ finite}\}.$$

Example 6.8. The elements of $\mathbf{I}_\omega(K_L;A)$ are precisely the finite closed sets of $\mathbf{SP}(A)$, since for any finite set B_0 of posets in $\mathbf{SP}(A)$, $cl(B_0)$ is finite.

Corollary 6.9. *Suppose that B_0 and C_0 are finite subsets of $F_K(A)$. Then*

$$cl(B_0) \subseteq cl(C_0)$$

iff for each ordered bimonoid homomorphism $h : F_K(A) \to \mathbf{SO}$ with $S \in K$ we have

$$B_0 h \leqslant C_0 h.$$

Thus

$$cl(B_0) = cl(C_0)$$

iff for each ordered bimonoid homomorphism $h : F_K(A) \to \mathbf{SO}$ with $S \in K$ we have

$$B_0 h = C_0 h.$$

Proof. Suppose first that $cl(B_0) \subseteq cl(C_0)$ and that $h : F_K(A) \to \mathbf{SO}$ is an ordered bimonoid morphism with $S \in K$. Since each $b \in B_0$ belongs to $cl(C_0)$, $bh \leqslant C_0 h$. Hence, $B_0 h = \sup_{b \in B_0} bh \leqslant C_0 h$. The other direction is immediate. □

Example 6.10. In the case of $\mathbf{SP}(A)$, $cl(B_0) \subseteq cl(C_0)$ iff $B_0 h_0 \subseteq C_0 h_0$ iff for each (maximal) word $u \in B_0 h_0$ there is a (maximal) word $v \in C_0 h_0$ with $u \sqsubseteq v$.

Definition 6.11. Suppose that $B, C \in \mathbf{I}_\omega(K;A)$. Then we define:

$$B + C := cl(B \cup C)$$

$$B \cdot C := cl(\{b \cdot c : b \in B,\ c \in C\})$$

$$B \otimes C := cl(\{b \otimes c : b \in B,\ c \in C\})$$

$$0 := \emptyset$$

$$1 := cl(\{1\}).$$

Proposition 6.12. *If $B, C \in \mathbf{I}_\omega(K;A)$ then $B + C$, $B \cdot C$, $B \otimes C$ are in $\mathbf{I}_\omega(K;A)$. If $B = cl(B_0)$ and $C = cl(C_0)$, with B_0 and C_0 finite, then*

$$B + C = cl(B_0 \cup C_0)$$

$$B \cdot C = cl(\{b \cdot c : b \in B_0,\ c \in C_0\})$$

$$B \otimes C = cl(\{b \otimes c : b \in B_0,\ c \in C_0\}).$$

Proof. We prove only one of the last three claims. Since $B_0 \cup C_0 \subseteq B + C$ and $B + C$ is closed, it follows that $cl(B_0 \cup C_0) \subseteq B + C$. Now assume that $x \in B + C$. Then there is a finite subset X_0 of $B \cup C$ which causes $x \in B + C$. But each element of X_0 is in turn caused by a (finite) subset of $B_0 \cup C_0$, and hence x is caused by $B_0 \cup C_0$. □

Let

$$\begin{aligned}\eta : A &\to \mathbf{I}_\omega(K;A)\\ a &\mapsto cl(\{a\}).\end{aligned}$$

Example 6.13. When $K = K_L$ so that $F_K(A)$ is $\mathbf{SP}(A)$,

$$a\eta = cl(\{a\}) = \{a\},$$

for each $a \in A$.

Theorem 6.14. *For each function $h : A \to S$, $S \in K$, there exists a unique homomorphism*

$$h^\sharp : \mathbf{I}_\omega(K;A) \to S$$

such that $\eta \cdot h^\sharp = h$.

Proof. First extend h to an ordered bimonoid morphism $h : F_K(A) \to S\mathbf{O}$. The only possible definition of $h^\sharp$ is the following. For each $B = cl(B_0) \in \mathbf{I}_\omega(K;A)$ with B_0 finite, define

$$Bh^\sharp := B_0 h.$$

The function $h^\sharp$ is well-defined by Corollary 6.9. By Proposition 6.12, the function $h^\sharp$ preserves the operations $+$, $\cdot$ and $\otimes$. Moreover, $h^\sharp$ preserves the constants. □

Example 6.15. When $K = K_L$, $F(A) = \mathbf{SP}(A)$ and S is the shuffle semiring $\mathbf{L}_\Sigma$, $Bh^\sharp = \bigcup_{P\in B_0} Ph = B_0 h$.

Proposition 6.16. *$\mathbf{I}_\omega(K;A)$ is in $\mathscr{V}$.*

Proof. There exists a representative set of ordered bimonoid morphisms

$$h_i : F_K(A) \to S_i\mathbf{O}, \quad S_i \in K,\ i \in I,$$

such that for all finite sets $B \subseteq F_K(A)$ and for all $b \in A$ the following two conditions are equivalent:

- $bh \leqslant Bh$, for each ordered bimonoid morphism $h : F_K(A) \to S\mathbf{O}$, $S \in K$.
- $bh_i \leqslant Bh_i$, for each $i \in I$.

Each h_i can be extended to a homomorphism

$$h_i^\sharp : \mathbf{I}_\omega(K;A) \to S_i.$$

Thus we obtain a homomorphism

$$\begin{aligned}h^\sharp : \mathbf{I}_\omega(K;A) &\to \prod_{i\in I} S_i\\ B &\mapsto (h_i^\sharp(B))_{i\in I}.\end{aligned}$$

But $h^\sharp$ is injective. It follows that $\mathbf{I}_\omega(K;A)$ is a shuffle semiring and belongs to the variety $\mathbf{Var}[K]$. $\square$

Example 6.17. In the case $K = K_L$, the representative set of morphisms in the proof of Proposition 6.16 is a singleton: $h_0^\# : \mathbf{I}_\omega(K_L;A) \to \mathbf{L}_{\Sigma(A)}$ which takes B to Bh_0.

Corollary 6.18. *The shuffle semiring $\mathbf{I}_\omega(K;A)$ is freely generated by A in $\mathbf{Var}[K]$.*

Proof. This is a restatement of Theorem 6.14 and Proposition 6.16. $\square$

Corollary 6.19. *When $K = K_L$, $\mathbf{I}_\omega(K_L;A)$ is freely generated by A in* **Lang**.

Remark 6.20. By Example 6.17, $\mathbf{I}_\omega(K_L;A)$ is isomorphic to a sub-shuffle semiring of $\mathbf{L}_{\Sigma(A)}$, denoted

$$H_\omega(A).$$

Indeed, $H_\omega(A)$ is the least sub-shuffle semiring of $\mathbf{L}_{\Sigma(A)}$ containing the sets ah_0, $a \in A$. Indeed, if $B \subseteq \mathbf{SP}(A)$ is a closed set, $Bh_0 = \sum_{P \in B} Ph_0$, and each set Ph_0 is obtainable from the bimonoid operations applied to the sets ah_0, $a \in A$.

Remark 6.21. Note that the above argument shows the following. If $V \subseteq W$ are varieties of shuffle semirings such that $V\mathbf{O} = W\mathbf{O}$, then $V = W$. (This fact follows also from the observation that if S, S' are shuffle semirings such that $S\mathbf{O} = S'\mathbf{O}$, then $S = S'$. The addition operation is determined by the ordering, since $a+b = \sup\{a,b\}$.) Thus, we can strengthen Corollary 6.18. Let W denote the collection of all shuffle semirings S such that $S\mathbf{O}$ belongs to the variety generated by $K\mathbf{O}$. It is not hard to show W is a variety, and clearly $\mathbf{Var}[K] \subseteq W$. Hence $\mathbf{Var}[K] = W$.

Remark 6.22. One can show that the variety **Lang** is also generated by either the shuffle semirings of finite languages, or those of regular languages, using extensions of the method used in Theorem 4.15. Again, using Remark 4.16, it follows that the equational theory of this variety is decidable.

6.2. Free complete shuffle semirings

A variety of complete shuffle semirings is a collection closed under products, homomorphic images and substructures, where homomorphisms preserve the infinitary operations as well. There is a functor $\mathbf{O}_\infty$ from the category $\mathbf{S}_\infty$ of all complete shuffle semirings to ordered bimonoids.

Suppose now that K is a class of complete shuffle semirings and $\mathbf{Var}_\infty[K]$ is the variety of complete shuffle semirings generated by K. Let $\mathscr{V}$ be the variety of ordered bimonoids generated by the class $K\mathbf{O}_\infty$. Let $F_K(A)$ be the ordered bimonoid freely generated by A in $\mathscr{V}$. By generalizing the notion of "closed" subset, we find a representation for the free algebras in $\mathscr{V}_\infty$.

Definition 6.23. We let **CLang** denote the variety generated by the class K_L, considered as a class of complete shuffle semirings.

Definition 6.24. Let $h : F_K(A) \to S\mathbf{O}_\infty$ be an ordered bimonoid morphism with $S \in K$. For any subset B of $F_K(A)$ define

$$Bh := \sum_{b \in B} bh.$$

For any subset B of $F_K(A)$ let $Cl(B)$ denote the set of all $b \in F_K(A)$ such that

$$bh \leqslant Bh,$$

for all morphisms $h : F_K(A) \to S\mathbf{O}$, $S \in K$.

Definition 6.25. A subset B of $F_K(A)$ is *completely closed* if $B = Cl(B)$. We let $\mathbf{I}(K;A)$ denote the collection of all completely closed subsets of $F_K(A)$.

Example 6.26. A subset B of $\mathbf{SP}(A)$ is completely closed iff B is closed iff $P \in B$ whenever $Ph_0 \subseteq \bigcup_{Q \in B} Qh_0$.

Each subset X of $F_K(A)$ is contained in a least completely closed subset, namely $Cl(X)$. Note that if X is finite, $Cl(X) = cl(X)$. We define complete shuffle semiring operations on the completely closed subsets by extending Definition 6.11 by:

Definition 6.27.

$$\sum_{i \in I} B_i := Cl\left(\bigcup_{i \in I} B_i\right),$$

and otherwise replacing cl by Cl.

Using the same argument as that for Theorem 6.18, we can prove the following fact.

Theorem 6.28. *For each set A, the complete shuffle semiring $\mathbf{I}(K;A)$ is freely generated by A in $\mathbf{Var}_\infty[K]$.*

Corollary 6.29. *When $K = K_L$, $\mathbf{I}(K_L;A)$ is the complete shuffle semiring in* **CLang** *freely generated by A.*

Remark 6.30. Using Example 6.17 again, it follows that $\mathbf{I}(K_L;A)$ is isomorphic to a complete sub-shuffle semiring $H(A)$ of $\mathbf{L}_{\Sigma(A)}$.

6.3. *Free star shuffle semirings*

A last application of closed sets is to the variety $\mathbf{Lang}_*$ generated by all star shuffle semirings

$$\mathbf{L}_\Sigma^* := (P_\Sigma, +, \cdot, \otimes, {}^*, 0, 1),$$

where, for a language $a \subseteq \Sigma^*$,

$$a^* := 1 + a + a^2 + \cdots .$$

Suppose that B is a completely closed set in $\mathbf{I}(K_L; A)$, the shuffle semiring of all completely closed subsets of $\mathbf{SP}(A)$. Then define B^* as the infinite sum

$$\sum_{n=0}^{\infty} B^n,$$

i.e., the closure of the union $\bigcup_{n \geqslant 0} B^n$. Then $\mathbf{I}(K_L; A)$ becomes a star shuffle semiring.

The following fact follows immediately from Remark 6.30.

Proposition 6.31. *For any $B \in \mathbf{I}(K_L; A)$,*

$$(B^*)h_0 = (Bh_0)^*.$$

Let $\mathbf{I}_*(A) \subseteq \mathbf{I}(K_L; A)$ be the least sub-star shuffle semiring of $I(K_L; A)$ consisting of the least set of completely closed subsets of $\mathbf{SP}(A)$ containing the principal ideals, closed under the shuffle semiring operations as well as under $B \mapsto B^*$.

Proposition 6.32. *$\mathbf{I}_*(A)$ belongs to the variety* $\mathbf{Lang}_*$.

Proof. It follows from Proposition 6.31 and the earlier examples that $\mathbf{I}_*(A)$ is isomorphic to a sub-star shuffle semiring of languages on $\Sigma(A)$. □

Theorem 6.33. *Let h be any function from A to $\mathbf{L}_\Sigma^*\mathbf{O}$, the underlying ordered bimonoid of a star shuffle semiring of languages. Then there is a unique star shuffle semiring morphism $h^\# : \mathbf{I}_*(A) \to \mathbf{L}_\Sigma^*$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "h"'] & \mathbf{I}_*(A) \arrow[d, "h^\#"] \\ & \mathbf{L}_\Sigma^* \end{tikzcd}$$

commutes, where $a\eta = cl(\{a\}) = Cl(\{a\})$. Thus, $\mathbf{I}_*(A)$ is freely generated by A in $\mathbf{Lang}_*$.

Proof. To obtain $h^\#$, first extend h to a complete shuffle semiring morphism $\mathbf{I}(K_L; A) \to \mathbf{L}_\Sigma$, and then restrict it to $\mathbf{I}_*(A)$. □

Corollary 6.34. *$\mathbf{I}_*(A)$ is freely generated by the set A in the variety generated by all star shuffle semirings*

$$\mathbf{R}_\Sigma^* := (\mathscr{R}_\Sigma, +, \cdot, \otimes, {}^*, 0, 1).$$

Proof. If $Ph_0 \subseteq (Q_1 \otimes Q_2)h_0$, then the width of P is at most the sum of the widths of the Q_i. Since serial product does not increase widths, the width of any completely closed set in $\mathbf{I}_*(A)$ is finite. Using this fact, one can show that $\mathbf{I}_*(A)$ is isomorphic to a substructure of a product of shuffle semirings $(\mathscr{R}_\Sigma, +, \cdot, \otimes, ^*, 0, 1)$, as in Theorem 4.15. The result follows. □

We omit the proof of the last result.

Corollary 6.35. *The variety* $\mathbf{Lang}_*$ *of shuffle semirings generated by all structures* $\mathbf{L}_\Sigma^*$ *is the same as that generated by just the regular ones* $\mathbf{R}_\Sigma^*$. *Moreover, the equational theory of this variety is decidable.*

Remark 6.36. Using Example 6.17 again, it follows that $\mathbf{I}_*(A)$ is isomorphic to a sub-star shuffle semiring $H^*(A)$ of $\mathbf{L}_{\Sigma(A)}$.

Remark 6.37. It seems clear that our methods extend without difficulty to finding the free algebras in the variety generated by the language star shuffle semirings enriched by the "iterated shuffle" operation: $B^{\otimes} := 1 + B + (B \otimes B) + (B \otimes B \otimes B) + \cdots$.

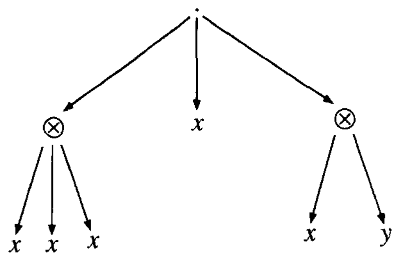
7. Complexity

A star shuffle semiring term over a countable set $X = \{x_1, x_2, \ldots\}$ of variables is defined in the usual way. A shuffle semiring term is a star shuffle semiring term not containing the symbol *, and a bimonoid term is a shuffle semiring term in which the symbols $+$ and 0 do not occur. The length of the term t, denoted $|t|$, is the total number of symbols occurring in t. Suppose that t and t' are terms. The length of an equation $t = t'$, or inequation $t \leqslant t'$, is $|t| + |t'| + 1$.

Due to the fact that $\mathbf{Lg}$ is the variety of all bimonoids, there is an $O(n \log n)$ time algorithm to decide if an equation $t = t'$ between the bimonoid terms t and t' holds in $\mathbf{Lg}$, where n denotes the length of the equation. The algorithm works in the following way.

First, there is a linear algorithm to check if the equation $t = 1$ holds in $\mathbf{Lg}$, in notation: $\mathbf{Lg} \models t = 1$. Moreover, when $\mathbf{Lg} \not\models t = 1$, the algorithm produces a bimonoid term t_1 which contains no occurrence of 1 and such that $\mathbf{Lg} \models t = t_1$. Thus, we may assume that the symbol 1 does not occur in the terms t and t'. In the second phase, we transform the terms t and t' to rooted, directed, vertex labeled trees $\hat{t}$ and $\hat{t'}$, by a linear algorithm. The vertex labels are in the set $X \cup \{\cdot, \otimes\}$. The outgoing edges of each vertex labeled $\cdot$ are linearly ordered. Moreover, if a vertex is labeled by $\otimes$, then its successors are labeled by symbols in the set $X \cup \{\cdot\}$; and if a vertex is labeled by $\cdot$, then its successors are labeled by symbols in $X \cup \{\otimes\}$. It is known that the isomorphism of such trees can be checked by an $O(n \log n)$ algorithm [12]. We omit the details of constructing the trees $\hat{t}$ and $\hat{t'}$. When t is the term $(x \otimes x \otimes x) \cdot (x \cdot (y \otimes x))$,

$\hat{t}$ is the tree shown below.



By the results of Section 4, the problem of deciding whether an inequation $t \leqslant t'$ holds in $\mathbf{Lg}_{\leqslant}$ is polynomial time reducible to the problem of deciding if $P \leqslant P'$ holds in the trace ordering for given A-labeled series-parallel posets P, Q, and vice versa. The complexity of a related problem, the language containment problem for pomsets, is studied in [7]. We formulate the problem to involve only series-parallel posets.

Definition 7.1. The language $L(P)$ of a poset $P \in \mathbf{SP}(A)$ is the set Ph, where h is the bimonoid morphism $\mathbf{SP}(A) \to \mathscr{L}_A$ with $ah = \{a\}$, for each $a \in A$.

Note that this definition of $L(P)$ is not the language used just above Lemma 5.13.

Lemma 7.2. *Suppose that $P, Q \in \mathbf{SP}(A)$. If $P \leqslant Q$ then $L(P) \subseteq L(Q)$.*

The lemma is an immediate corollary of Theorem 5.16.

The language containment problem for series-parallel posets is the problem of deciding whether $L(P) \subseteq L(Q)$ holds for given series-parallel posets P and Q. It is shown in [7] that the language containment problem for *arbitrary* labeled posets is Π_2^p-complete. The proof is by reduction from the bounded B_2^c problem that we recall now.

Input: Two sets $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ of Boolean variables and a set $C = \{c_1, \ldots, c_k\}$ of clauses $a \Rightarrow b \vee c \vee d$, where a is x_i or x_i', the Boolean complement of x_i, for some $i \in [m]$, and each of b, c and d is either y_i or y_i', for some $i \in [n]$.

Question: Is it the case that for every truth assignment to the variables in X there is a truth assignment to the variables in Y such that each clause in C is satisfied?

Notation: A clause c of the form $a \Rightarrow b_1 \vee b_2 \vee b_3$ will be called a clause implied by the literal a, denoted $c \in \operatorname{imp}(a)$.

Theorem 7.3. *The problem of deciding whether $P \leqslant Q$ holds for series-parallel posets $P, Q \in \mathbf{SP}(A)$ is Π_2^p-complete.*

Remark 7.4. A different proof of this result can be found in [13].

Corollary 7.5. *Deciding the inequational theory of the variety $\mathbf{Lg}_{\leqslant}$ is Π_2^p-complete.*

Proof. If R is a poset in $\mathbf{SP}(A)$, let width(R) denote the width of R. Call a poset R' a normal relabeling of R if R' is obtained from R by relabeling the vertices labeled a, for each $a \in A$, by words $a_i\bar{a}_i$ such that $1 \leqslant i \leqslant \mathrm{width}(R)$ and such that any two incomparable vertices are labeled differently. Recall the concept of an expansion of a labeled poset.

The fact that $P \leqslant Q$ holds for $P, Q \in \mathbf{SP}(A)$ can be expressed by the following predicate which shows the problem is in Π_2^p: For each normal relabeling P' of P, and for each linearization of the expansion of P' (determined by the relabeling), with associated trace $u \in A^*$, there exists some relabeling Q' of Q such that some linearization of the expansion of Q' has associated trace the word u.

In the rest of the proof, we indicate how the argument proving Theorem 3.1 in [7] can be modified to obtain Π_2^p-hardness. Suppose that (X, Y, C) is an instance of B_2^c. We construct two posets P and Q in $\mathbf{SP}(A)$, where A is the set $X \cup Y \cup C$. (We assume these sets are disjoint.)

First, for each $x_i \in X$, let

$$X_i := x_i \cdot \bigotimes(c_j : c_j \in \mathrm{imp}(x_i))$$
$$X_i' := x_i \cdot \bigotimes(c_j : c_j \in \mathrm{imp}(x_i')).$$

Then, for each $y_i \in Y$, let

$$Y_i := y_i \cdot \bigotimes(c_j : y_i \text{ occurs in } c_j)$$
$$Y_i' := y_i \cdot \bigotimes(c_j : y_i' \text{ occurs in } c_j).$$

Here, we understand that Y_i (respectively Y_i') contains a vertex labeled c_j for each *occurrence* of y_i (respectively y_i') in c_j. Finally, for each $x_i \in X$, let

$$Z_i := (x_i \otimes x_i) \cdot \bigotimes(c_j : c_j \in \mathrm{imp}(x_i) \cup \mathrm{imp}(x_i')).$$

Then we define:

$$P := y_1 \cdot \ldots \cdot y_n \cdot \bigotimes_{i=1}^{m}(X_i \otimes X_i') \cdot y_1 \cdot \ldots \cdot y_n \cdot c_1^3 \cdot \ldots \cdot c_k^3$$
$$Q := \bigotimes_{j=1}^{n}(Y_j \otimes Y_j') \otimes \bigotimes_{i=1}^{m} Z_i.$$

Thus, the poset Q is the same as the second poset, but P is a modified version of the first poset constructed in the proof of Theorem 3.1 in [7], which is not series-parallel.

Lemma 7.6. *The following conditions are equivalent*:
1. $P \leqslant Q$;
2. $L(P) \subseteq L(Q)$;
3. (X, Y, Z) *is a YES instance of* B_2^c.

The proof of this lemma is essentially the same as the argument given in [7]. □

The problem of deciding the equational theory of the variety **Lang** is easily seen to be also in Π_2^p. But if t and t' are bimonoid terms, then $\mathbf{Lg}_\leqslant \models t \leqslant t'$ iff $\mathbf{Lang} \models t+t' = t'$. Thus we have:

Corollary 7.7. *The problem of deciding the equational theory of the variety* **Lang** *is* Π_2^p*-complete.*

Concerning the variety $\mathbf{Lang}_*$, we mention the following theorem of Meyer [14].

Theorem 7.8. *The problem of deciding the equational theory of the variety* $\mathbf{Lang}_*$ *is* EXPSPACE*-complete.*

8. Some remarks

After the free ordered bimonoids were found in the class $\mathbf{Lg}_\leqslant$, the constructions of Sections 6.1, 6.2, and 6.3 were forced. What were not routine were the definition of the ordering on $\mathbf{SP}(A)$, and Proposition 5.11 showing that the ordering on the posets in $\mathbf{SP}(A)$ is the correct one.

We have given two kinds of descriptions of the free algebras in varieties of bimonoids and shuffle semirings of various kinds: as (closed subsets of) labeled posets ($\mathbf{SP}(A)$, equipped with various operations and orderings) and as sets of words (see Definition 4.14 and Remarks 6.20, 6.30, and 6.36). The main idea was to separate distinct A-labeled posets by maximal traces of their expansions, obtained by replacing the letter $a \in A$ in all possible ways by the words $a_i\bar{a}_i$, $i \geqslant 0$. (See the Open Problems, below.) There is at least one alternative to this particular construction.

One may use instead of maximal traces, only the "conservative traces" of the expansion. (A word u in Ph_0 is conservative if for each $a_i \in A_N$ and each pair of words u_1, u_2, if $u = u_1 a_i u_2$, then a_j is open in u_1, for each $j < i$, but a_i is not.) A conservative trace does not require distinct words $a_i\bar{a}_i$, for all vertices labeled $a \in A$, but only as many as the width of the poset, i.e., at most the maximum number of nodes that may be open at one time. (Recall Remark 4.16.) Since the Kleene *-operation does not increase width, only finite languages are needed to separate the (ideals of) posets generated by the regular operations, together with shuffle. This result was also found by Meyer and Rabinovich [15], and a related result was proved in [9]. It follows that the equational theory of $\mathbf{Lang}_*$ is decidable.

In a forthcoming paper [5], the authors show that the variety $\mathbf{Lg}_\leqslant$ is not finitely axiomatizable. This is in contrast to the variety **Lg** which is finitely axiomatizable. In [6], it is shown that the varieties generated by the language structures $(P_\Sigma, \cdot, \otimes, +, 0, 1)$ and $(P_\Sigma, \cdot, \otimes, +, ^*, 0, 1)$ are also not finitely axiomatizable.

9. Open problems

1. In the proof of Theorem 4.15, for $n \geqslant 1$, the bimonoid morphism $h_n : \mathbf{SP}(A) \to \mathscr{F}_{\Sigma(A)}$ was defined as the unique bimonoid morphism mapping the letter $a \in A$ to the finite language $\{a_i\bar{a}_i : i = 0, \ldots, n-1\}$. The question is whether there is one value of $n > 0$ such that for all posets $P, Q \in \mathbf{SP}(A)$,

$$Ph_n \subseteq Qh_n \Rightarrow P \leqslant Q.$$

(Note that $P \leqslant Q \Rightarrow Ph_n \subseteq Qh_n$, all $n \geqslant 1$.) For example, define P, Q as follows:

$$P := a \otimes (a \cdot a \cdot a)$$
$$Q := (a \cdot a) \otimes (a \cdot a).$$

Then $Ph_1 \subseteq Qh_1$, but it is not the case that $P \leqslant Q$. A weaker related question is whether there is some $n \geqslant 1$ such that for all $P, Q \in \mathbf{SP}(A)$,

$$Ph_n = Qh_n \quad \Rightarrow \quad P = Q.$$

In [1], it was conjectured that $n = 1$ suffices.

2. Find a characterization of the languages $\mathbf{Sh}_A$, $H_\omega(A)$, $H(A)$ and $H^*(A)$ representing the various free algebras. These languages were defined in Remarks 4.14, 6.20, and 6.30.

3. Is the equational theory of star shuffle semirings with iterated shuffle decidable? (See Remark 6.37.)

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Appendix A. More on series-parallel posets

In this appendix, we give our proof of the main result of Gischer [10], that $\mathbf{SP}(A)$ equipped with the subsumption ordering $\preceq$ (defined in Remark 5.9) is freely generated by A in the variety of ordered bimonoids satisfying the weak interchange law. Then we prove the characterization of the series-parallel posets mentioned in Remark 3.32.

First, we note some simple properties of the series-parallel posets. The first lemma is proved by induction on the structure of the poset P.

Lemma A.1. *Suppose that P is a series-parallel A-labeled poset and that v is a vertex of P. Then the set $Q = P - \{v\}$, equipped with ordering and labeling inherited from P (i.e., for all vertices v_1 and v_2 other than v, $v_1 \leqslant_Q v_2$ iff $v_1 \leqslant_P v_2$), is also series-parallel.*

Corollary A.2. *Suppose that P is a series-parallel A-labeled poset. Suppose that Q is a subset of P. Then the set Q, equipped with the ordering and labeling inherited from P, is series-parallel.*

Corollary A.3. *Suppose that P is a series-parallel A-labeled poset. If $P = P_1 \cdot P_2$, or if $P = P_1 \otimes P_2$, for some A-labeled posets P_1 and P_2, then P_1 and P_2 are series-parallel.*

Recall that a poset, considered as a directed graph, is *connected* if the underlying undirected graph is. An A-labeled poset P is called *$\otimes$-indecomposable* if P is not the shuffle product $P_1 \otimes P_2$ of two nonempty A-labeled posets P_1 and P_2.

Lemma A.4. *Suppose that P is an A-labeled poset. Then P is connected iff P is $\otimes$-indecomposable. Thus, if P is series-parallel, then P is connected iff P is empty, a singleton, or P is the serial product of two nonempty series-parallel A-labeled posets.*

A.1. $\mathbf{SP}(A)$ *as a free ordered bimonoid*

We characterize the variety **WI** generated by the ordered bimonoids $(\mathbf{SP}(A), \preceq)$ by an inequation. Theorem A.9 below identifies the ordered bimonoid $(\mathbf{SP}(A), \preceq)$ as the free ordered bimonoid in **WI** generated by the set A.

Definition A.5. Suppose that P and Q are A-labeled posets. We define $P \preceq Q$, the *subsumption* or *less structure* ordering, if P and Q have the same underlying set, the same labeling, and

$$v \leqslant_Q v' \quad \Rightarrow \quad v \leqslant_P v',$$

for any two vertices $v, v' \in P$.

Since we have identified isomorphic A-labeled posets, the condition that P and Q have the same underlying set may be rephrased by requiring that there is a bijective labeled poset morphism $Q \to P$.

Proposition A.6. *The relation $\preceq$ is a partial order and the operations of serial and shuffle product monotonic. Thus,* $(\mathbf{Pos}(A), \preceq)$ *is an ordered bimonoid.*

Restricting the partial order $\preceq$ to the series-parallel posets, we obtain the ordered bimonoid $(\mathbf{SP}(A), \preceq)$.

Proposition A.7. *The inequation*

$$(P_1 \otimes P_2) \cdot (Q_1 \otimes Q_2) \preceq P_1 \cdot Q_1 \otimes P_2 \cdot Q_2$$

holds for all A-labeled posets P_i and Q_i, $i = 1,2$.

Before presenting Theorem A.9, we establish an elementary property of A-labeled posets in conjunction with the $\preceq$ ordering.

Lemma A.8. *Suppose that $Q = Q_1 \otimes Q_2$ is the shuffle of two $\otimes$-indecomposable series-parallel A-labeled posets Q_1 and Q_2. Suppose that P is a series-parallel A-labeled poset with the same set of vertices such that $P \preceq Q$. If there are vertices $v_i \in Q_i$, $i = 1,2$, which are comparable in P, then P is the serial product of two nonempty series-parallel A-labeled posets P_1 and P_2.*

Proof. Indeed, if some $v_1 \in Q_1$ and $v_2 \in Q_2$ are comparable in P, then P is connected and hence P is the serial product of two nonempty series-parallel A-labeled posets P_1 and P_2. □

Theorem A.9. $(\mathbf{SP}(A), \preceq)$ *is the ordered algebra freely generated by the set A in the class of ordered bimonoids satisfying the* weak interchange law, *i.e., the inequation*

$$(a \otimes b) \cdot (x \otimes y) \leqslant ax \otimes by. \tag{A.1}$$

Proof. By Propositions A.6 and Proposition A.7. $(\mathbf{SP}(A), \preceq)$ is an ordered bimonoid satisfying the inequation (A.1).

To establish the universal property of $(\mathbf{SP}(A), \preceq)$, suppose that $(\mathrm{M}, \leqslant)$ is an ordered bimonoid satisfying the inequation (A.1). Let φ be a map $A \to M$. There is a unique extension of φ to a bimonoid morphism $\varphi^\# : \mathbf{SP}(A) \to \mathrm{M}$, by Theorem 3.3. We need to show that $\varphi^\#$ respects the ordering, i.e.,

$$P \preceq Q \;\Rightarrow\; P\varphi^\# \leqslant Q\varphi^\#, \tag{A.2}$$

for all series-parallel A-labeled posets P and Q. We prove this fact by induction on the number of vertices of Q. (We assume that P and Q have the same underlying set of vertices and that if $v \leqslant_Q v'$, for some vertices v and v', then $v \leqslant_P v'$.)

When Q is empty or has a single vertex, we have $P = Q$. Thus, $P\varphi^\# = Q\varphi^\#$.

Suppose that Q has more than one vertex. Let $n = v(P,Q)$ denote the number of ordered pairs (v, v') of vertices such that $v \leqslant_P v'$ but v and v' are not comparable in Q. When $n = 0$, we have $P = Q$, so that $P\varphi^\# = Q\varphi^\#$. Suppose that $n > 0$ and that we have proved that $P'\varphi^\# \leqslant Q'\varphi^\#$ for all series-parallel posets P' and Q' with $P' \preceq Q'$ which have the same number of vertices as Q and for which $v(P', Q') < n$. We consider two cases.

Case 1: Q is a serial product $Q_1 \cdot Q_2$ of two nonempty A-labeled series-parallel posets Q_1 and Q_2. Since $v_1 <_Q v_2$ for all vertices $v_i \in Q_i$, $i = 1,2$, and since $P \leqslant Q$, it follows that P can be written as a serial product $P_1 \cdot P_2$, for some A-labeled posets

P_i with $P_i \leqslant Q_i$, $i = 1,2$. By Corollary A.3, the posets P_i are series-parallel. But then,

$$\begin{aligned} P\varphi^\sharp &= P_1\varphi^\sharp \cdot P_2\varphi^\sharp \\ &\leqslant Q_1\varphi^\sharp \cdot Q_2\varphi^\sharp \\ &= Q\varphi^\sharp, \end{aligned}$$

since the series-parallel posets Q_i have fewer vertices than Q and since (M, $\leqslant$) is an ordered bimonoid.

Case 2: Q is the shuffle of two or more nonempty series-parallel A-labeled posets. Let us assume that $Q = Q_1 \otimes Q_2$, where Q_1 and Q_2 are $\otimes$-indecomposable series-parallel A-labeled posets. If there exist no vertices $v_i \in Q_i$, $i = 1,2$, which are comparable in P, then P is of the form $P = P_1 \otimes P_2$ for some A-labeled posets P_i with $P_i \preceq Q_i$, $i = 1,2$. By Corollary A.3, the posets P_i are series-parallel. Since the posets Q_i have fewer vertices than Q, we have

$$\begin{aligned} P\varphi^\sharp &= P_1\varphi^\sharp \otimes P_2\varphi^\sharp \\ &\leqslant Q_1\varphi^\sharp \otimes Q_2\varphi^\sharp \\ &= Q\varphi^\sharp, \end{aligned}$$

by the induction assumption and since the operation $\otimes$ is monotonic in B.

Suppose now that there exists a pair of vertices $v_1 \in Q_1$ and $v_2 \in Q_2$ which are comparable in P. Then, by Lemma A.8, P is a serial product of nonempty series-parallel A-labeled posets, say $P = R \cdot S$. Let R_i, $i = 1,2$, be the A-labeled sub-poset of P determined by the vertices in $Q_i \cap R$. Similarly, let S_i, $i = 1,2$, be the A-labeled sub-poset of P determined by the vertices in $Q_i \cap S$. Thus, for any two vertices v and v' in $Q_i \cap R$, $v \leqslant_{R_i} v'$ iff $v \leqslant_P v'$. By Corollary A.2, the posets R_i and S_i are series-parallel. We have the following inequations:

$$P \preceq R_1 \cdot S_1 \otimes R_2 \cdot S_2 \tag{A.3}$$

$$R_i \cdot S_i \preceq Q_i, \quad i = 1,2. \tag{A.4}$$

Let $P' = R_1 \cdot S_1 \otimes R_2 \cdot S_2$. Since $\nu(P,P') < n$,

$$P\varphi^\sharp \leqslant (R_1 \cdot S_1 \otimes R_2 \cdot S_2)\varphi^\sharp, \tag{A.5}$$

by the induction assumption and (A.3). Since for $i = 1,2$, the number of vertices of Q_i is strictly less than the number of vertices of Q, we have also

$$(R_i \cdot S_i)\varphi^\sharp \leqslant Q_i\varphi^\sharp, \tag{A.6}$$

by (A.4). Thus,

$$\begin{aligned} P\varphi^\sharp &\leqslant (R_1 \cdot S_1 \otimes R_2 \cdot S_2)\varphi^\sharp \\ &= (R_1 \cdot S_1)\varphi^\sharp \otimes (R_2 \cdot S_2)\varphi^\sharp \\ &\leqslant Q\varphi^\sharp, \end{aligned}$$

using (A.5), (A.6), the assumption that (M, $\leqslant$) is an ordered bimonoid satisfying (A.1), and the fact that $\varphi^\sharp$ is a bimonoid morphism. □

Remark A.10. We note that our proof of Theorem A.9 is not syntactic, as is the proof given by Gischer in [10].

Let **WI** denote the variety of ordered bimonoids which satisfy the weak interchange law (A.1).

Corollary A.11. *The following varieties of ordered bimonoids are the same*:
1. *The variety* **WI**.
2. *The variety generated by the ordered bimonoids* $(\mathbf{SP}(A), \preceq)$.
3. *The variety generated by the ordered bimonoids* $(\mathbf{Pos}(A), \preceq)$.

A.2. The geometry of series-parallel posets

We give a graph theoretic characterization of the posets in $\mathbf{SP}(A)$.

Recall that we have already shown that any poset in $\mathbf{SP}(A)$ has the zig-zag property.

Definition A.12. Suppose that P is an A-labeled poset. A *zig-zag* in P is a pair (L, U) of nonempty subsets of P such that for each $v \in L$ and $v' \in U$,

$$U = succ(v) \quad \text{and} \quad L = pred(v').$$

Note that if P has the zig-zag property, and if v is not a maximal vertex in P, then there is a unique zig-zag (L, U) with $v \in L$, namely $U = succ(v)$ and $L = pred(succ(v))$. Similarly, if v' is not minimal, then there is a unique zig-zag (L, U) with $v \in U$, namely $L = pred(v')$ and $U = succ(pred(v'))$.

Suppose that (L, U) is a zig-zag in P. Then for each vertex $x \in P$, if $x < v$ for some $v \in U$, then $x < v'$ for all $v' \in U$. In this case we also write $x < U$. If $v < x$ for some vertex $v \in L$, then $L < x$, i.e., $v' < x$ for all $v' \in L$. We define

$$P_{(L,U)} := \{x : x < U \text{ or } L < x\}.$$

Thus $P_{(L,U)}$ is the set of all elements comparable with some element in $L \cup U$. When (L_1, U_1) and (L_2, U_2) are both zig-zags, we write $L_1 < U_2$ if $x < y$ holds for some (or for all) $x \in L_1$ and $y \in U_2$.

Theorem A.13. *An A-labeled poset P is series-parallel iff it has the zig-zag property and whenever (L_1, U_1) and (L_2, U_2) are zig-zags with $L_1 < U_2$, there is a zig-zag (L, U) such that:*
(a) $L < U_2$ *and* $L_1 < U$.
(b) $P_{(L_1,U_1)} \cup P_{(L_2,U_2)} \subseteq P_{(L,U)}$.

Proof. We have already noted that each series-parallel poset has the zig-zag property. A straightforward induction argument proves that if P is series-parallel, then for any two zig-zags (L_1, U_1) and (L_2, U_2) with $L_1 < U_2$ there exists a zig-zag (L, U) such that the conditions (a) and (b) hold. Before proving the converse direction, we need a lemma.

Lemma A.14. *Suppose that $P = P_1 \cdot P_2$ or $P = P_1 \otimes P_2$, where P is an A-labeled poset having the zig-zag property and such that for any two zig-zags (L_1, U_1) and (L_2, U_2) with $L_1 < U_2$ there is some zig-zag (L, U) such that* (a) *and* (b) *hold. Then P_1 and P_2 also have these properties.c*

Proof of Theorem A.13 (*continued*). Suppose that P has the zig-zag property and satisfies the conditions (a) and (b) of the theorem, for any two zig-zags (L_1, U_1) and (L_2, U_2) with $L_1 < U_2$. We prove P is series-parallel by induction on the number of vertices. If P is empty or a singleton, then P is series-parallel. If P has more than one vertex, let

$$\{(L_1, U_1), \ldots, (L_k, U_k)\}$$

be a set of (distinct) zig-zags with least cardinality such that

$$\bigcup_{i=1}^{k} P_{(L_i, U_i)} = P, \tag{A.7}$$

i.e., each vertex in P is comparable with some vertex in some zig-zag (L_i, U_i), and, moreover,

$$max\{card(P_{(L_i,U_i)}) : i = 1, \ldots, k\} \geqslant max\{card(P_{(L_i',U_i')}) : i = 1, \ldots, k\},$$

whenever $\{(L_1', U_1'), \ldots, (L_k', U_k')\}$ is another set of zig-zags with the property (A.7). (Here, when A is a set, $card(A)$ denotes the cardinality of A.)

When $k = 1$, P is the serial product $P = R_1 \cdot R_2$, where

$$R_1 = \{x : x < U_1\}$$
$$R_2 = \{x : L_1 < x\}.$$

The vertices in R_1 and R_2 are ordered and labeled as in P. Now, the A-labeled posets R_1 and R_2 are series-parallel by Lemma A.14 and the induction hypothesis. It follows that P is series-parallel.

When $k > 1$, we show that P is the shuffle product of the disjoint sub-posets

$$P_i = P_{(L_i, U_i)}, \quad i = 1, \ldots, k.$$

It then follows from the induction assumption and Lemma A.14 that P is series-parallel. We assume that $k = 2$, since the argument is similar for $k > 2$.

Without loss of generality we may assume that $card(P_1) \geqslant card(P_2)$. We note that neither $L_1 < U_2$ nor $L_2 < U_1$ may hold, for otherwise we obtained a zig-zag (L, U)

with

$$P = P_1 \cup P_2 \subseteq P_{(L,U)}.$$

Thus the sets $L_1 \cup U_1$ and $L_2 \cup U_2$ are disjoint. Moreover there exists no vertex x with $L_2 < x < U_1$ or $L_1 < x < U_2$. Thus P_1 and P_2 have no vertex in common iff there is no vertex x with $x < U_1$ and $x < U_2$, or $L_1 < x$ and $L_2 < x$. Indeed, suppose that $x < U_1$ and $x < U_2$, say. Let (L_3, U_3) be the zig-zag with $x \in L_3$. Since $L_3 < U_1$, there exists a zig-zag (L, U) with $L_3 < U$, $L < U_1$ and

$$P_1 \cup P_{(L_3,U_3)} \subseteq P_{(L,U)}. \tag{A.8}$$

Thus $P_{(L,U)} \cup P_2 = P$.

Consider now the the zig-zags (L, U) and (L_2, U_2). If $y \in U_2$, then $x < y$, and thus $y \in P_{(L,U)}$, by (A.8). Thus, either $L < U_2$ or $U_2 < U$. If $U_2 < U$, then, by $L < U_1$, we have $L_2 < U_1$, which was shown to be impossible. Thus $L < U_2$. But then, there exists a zig-zag (L', U') with

$$P = P_{(L,U)} \cup P_2 \subseteq P_{(L',U')},$$

contrary to the assumption that there exists no zig-zag with this property.

We still need to show that if x and y are in the disjoint sets P_1 and P_2, respectively, then x and y are incomparable. By symmetry, the only nontrivial case to be considered is that $x < U_1$ and $L_2 < y$, and $y \in succ(x)$. Supposing this, let (L, U) be the zig-zag with $x \in L$ and $y \in U$. Applying the assumption for (L, U) and (L_1, U_1) we obtain a a zig-zag (L', U') with

$$P_{(L,U)} \cup P_1 \subseteq P_{(L',U')},$$

so that

$$P_{(L',U')} \cup P_2 = P.$$

But since $y \in P_{(L',U')}$, $card(P_{(L',U')}) > card(P_1)$, a contradiction. □

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